Matrix nearness problems for Lyapunov-type stability domains

computing Distance-to-Delocalization

Vladimir R. Kostić,
Faculty of Science, University of Novi Sad, Serbia

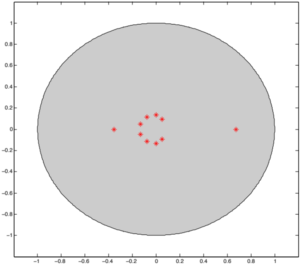
joint work with Agnieszka Międlar, Jeoren Stolwjik

SIAM LA 2015 | CP6 : October 27, 2015

Berlin Mathematical School

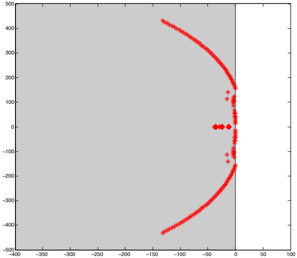
Stability of dynamical systems...

Stability of linear time invariant dynamical system at equilibrium point depends of the location of the system eigenvalues:



discrete case (d-stability)

$$x_{k+1} = Ax_k, k \in \mathbb{N}$$
$$\Lambda(A) \subseteq \mathcal{B}_1$$



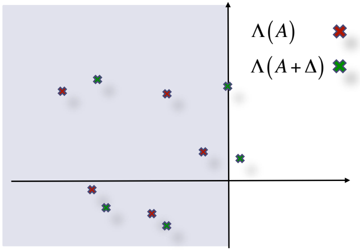
continuous case (c-stability)

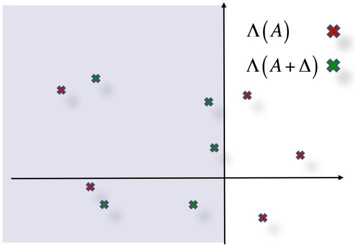
$$\dot{x}(t) = Ax(t), t \in \mathbb{R}_0^+$$
$$\Lambda(A) \subseteq \mathbb{C}^-$$

$$\Lambda(A) := \{z \in \mathbb{C} : \det(A - zI) = 0\}$$

Distance to c- instability/stability

$$\Lambda(A) := \{z \in \mathbb{C} : \det(A - zI) = 0\}$$





$\inf \|\Delta\|$

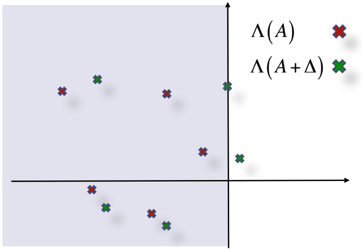
s.t. $\Lambda(A + \Delta) \not\subseteq \mathbb{C}^-$

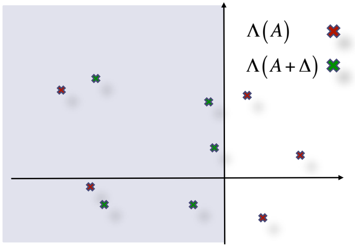
$\inf \|\Delta\|$

s.t. $\Lambda(A + \Delta) \subseteq \mathbb{C}^-$

Distance to c- instability/stability

$$\Lambda(A) := \{z \in \mathbb{C} : \det(A - zI) = 0\}$$





$$\inf \|\Delta\| \quad \text{s.t.} \ \Lambda(A+\Delta) \not\subseteq \mathbb{C}^-$$

$$\inf \|\Delta\| \quad \text{s.t.} \ \Lambda(A+\Delta) \subseteq \mathbb{C}^-$$

$$\Lambda_\varepsilon(A) := \bigcup_{\|\Delta\|<\varepsilon} \Lambda(A+\Delta)$$

L. N. Trefeten, M. Embree: *Spectra and Pseudospectra: The Behavior of Nonnormal Matrices and Operators*, Princeton University Press, 2005

Several references on the distance to instability problems:

R. Byers: *A Bisection Method for Measuring the Distance of a Stable Matrix to the Unstable Matrices*. SIAM J. Scientific and Statistical Computing, 9:875-881, 1988

C. He, G. A. Watson: *An algorithm for computing the distance to instability*. SIAM J. Matrix Analysis Applications 20:101-116, 1999

N. Guglielmi, M. L. Overton: *Fast Algorithms for the Approximation of the Pseudospectral Abscissa and Pseudospectral Radius of a Matrix*. SIAM J. Matrix Analysis Applications 32:1166-1192, 2011

M. Gurbuzbalaban, N. Guglielmi, M. L. Overton: *Fast Approximation of the H? Norm via Optimization over Spectral Value Sets*. SIAM J. Matrix Anal. Appl. 34 (2013), pp. 709-737

N. Guglielmi, D. Kressner, C. Lubich: *Low-rank diferential equations for Hamiltonian matrix nearness problems*. Oberwolfach-Walke : MFO, 2013

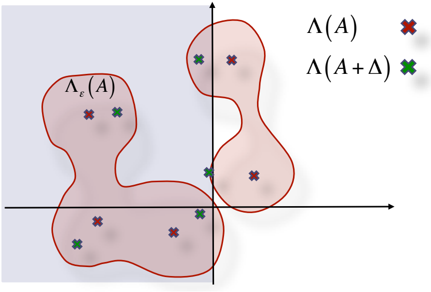
M. A. Freitag, A. Spence: *A Newton-based method for the calculation of the distance to instability*. Linear Algebra and Applications 435(12): 3189-3205, 2011

Distance to c-stability

Distance to c-stability

For a given A such that $\Lambda(A) \not\subseteq \mathbb{C}^-$ solve

$$\begin{aligned} &\inf \|\Delta\| \\ &\text{s.t. } \Lambda(A+\Delta) \subseteq \mathbb{C}^- \end{aligned}$$

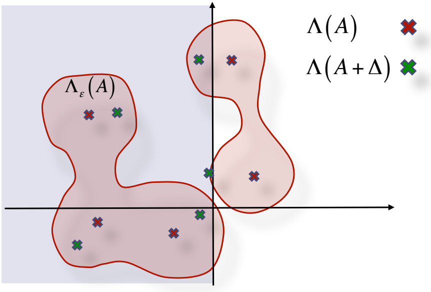


Distance to c-stability

For a given A such that $\Lambda(A) \not\subseteq \mathbb{C}^-$ solve

$$\inf \|\Delta\|$$
$$\text{s.t. } \Lambda(A+\Delta) \subseteq \mathbb{C}^-$$

Pseudospectral methods may not be the best choice!

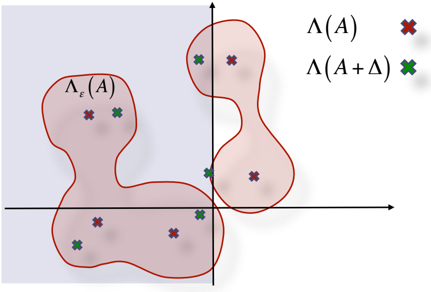


Distance to c-stability

For a given A such that $\Lambda(A) \not\subseteq \mathbb{C}^-$ solve

$$\inf \|\Delta\|$$
$$\text{s.t. } \Lambda(A+\Delta) \subseteq \mathbb{C}^-$$

Pseudospectral methods may not be the best choice!

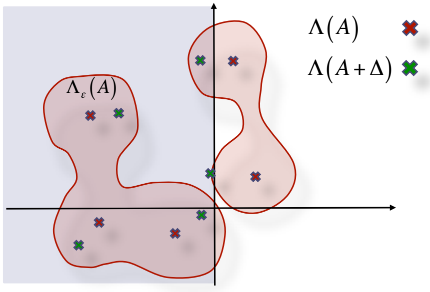


F.-X. Orbandexivry, Y. Nesterov, P. M. Van Dooren: *Nearest stable system using successive convex approximations*. Automatica 49: 1195-1203, 2011

Distance to c-stability

For a given A such that $\Lambda(A) \not\subseteq \mathbb{C}^-$ solve

$$\inf \|\Delta\| \quad \text{s.t. } \Lambda(A+\Delta) \subseteq \mathbb{C}^-$$



Pseudospectral methods may not be the best choice!

$$\min_{X,Y} \|X - A\|_F$$

$$\text{s.t. } -(XY + YX^*) \succ 0$$

$$Y \succ 0$$

Lyapunov stability test!

F.-X. Orbandexivry, Y. Nesterov, P. M. Van Dooren: *Nearest stable system using successive convex approximations*. Automatica 49: 1195-1203, 2011

WHAT ABOUT OTHER NOTIONS OF "STABILITY" ?!

Damped stability

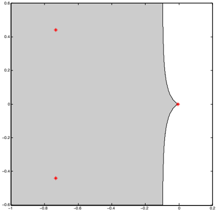
Not only that the system needs to be c-stable, but, additional constraints that depend on the imaginary parts of eigenvalues (frequency of the oscillations of the basic solutions) have to hold:

Damped stability

Not only that the system needs to be c-stable, but, additional constraints that depend on the imaginary parts of eigenvalues (frequency of the oscillations of the basic solutions) have to hold:

In the perturbed motion of a rocket, taking into account the elastic oscillations of its airframe as a straight flexible nonuniform rod:

the "stable" eigenvalues belong to the domain bounded by Cissoid of Diocles

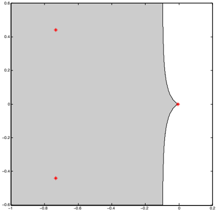


Damped stability

Not only that the system needs to be c-stable, but, additional constraints that depend on the imaginary parts of eigenvalues (frequency of the oscillations of the basic solutions) have to hold:

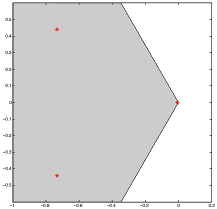
In the perturbed motion of a rocket, taking into account the elastic oscillations of its airframe as a straight flexible nonuniform rod:

the "stable" eigenvalues belong to the domain bounded by Cissoid of Diocles



Another damped stability domain that occurs in practice is a wedge around real axis:

the "stable" eigenvalues have complex arguments between $\pi - \theta$ and $\pi + \theta$



Frequency band of the undesirable noise

In structural acoustics, the localization of the eigenvalues in the complex plane corresponds to the appearance of acoustic waves of certain frequencies. In practice, certain frequency bands of the noise are of special interest.

Frequency band of the undesirable noise

In structural acoustics, the localization of the eigenvalues in the complex plane corresponds to the appearance of acoustic waves of certain frequencies. In practice, certain frequency bands of the noise are of special interest.

For example, the audible frequencies for an average human ear belong to the band 20Hz−20kHz, while in airplanes, vibrations below 10Hz have a profound influence on specific parts and systems of the human body.

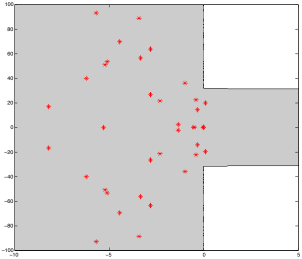
Frequency band of the undesirable noise

In structural acoustics, the localization of the eigenvalues in the complex plane corresponds to the appearance of acoustic waves of certain frequencies. In practice, certain frequency bands of the noise are of special interest.

For example, the audible frequencies for an average human ear belong to the band 20Hz−20kHz, while in airplanes, vibrations below 10Hz have a profound influence on specific parts and systems of the human body.

The system is not producing noise of the frequency between a_lHz and a_hHz when the spectrum is either

- in the left half-plane (stable modes), or
- out of the horizontal strips in the right half-plane $[-2\pi a_h, -2\pi a_l]$ and $[2\pi a_l, 2\pi a_h]$ (frequency region of unstable modes)



Stability of the robust reversible discrete process

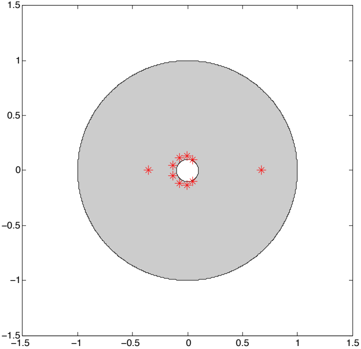
In the treatment of discrete dynamical systems, constrains additional to d-stability may exist, too.

Stability of the robust reversible discrete process

In the treatment of discrete dynamical systems, constrains additional to d-stability may exist, too.

In the Leslie model of the population dynamics, two main concerns are connected with the spectrum:

- existence of the stable equilibrium state (eigenvalues are in the open unit disk)
- the robust reversibility, i.e., the determinant is bounded away from zero (eigenvalues are out of the small open disk)



The goal

Having this in mind, our goal is to

The goal

Having this in mind, our goal is to

1. define the matrix nearness problems that cover more general notions of the "stability",

The goal

Having this in mind, our goal is to

1. define the matrix nearness problems that cover more general notions of the "stability",
2. develop appropriate tools for their solution,

The goal

Having this in mind, our goal is to

1. define the matrix nearness problems that cover more general notions of the "stability",
2. develop appropriate tools for their solution,
3. construct efficient computational techniques for practical use.

The goal

Having this in mind, our goal is to

1. define the matrix nearness problems that cover more general notions of the "stability",
2. develop appropriate tools for their solution,
3. construct efficient computational techniques for practical use.

Therefore, we are interested in:

The goal

Having this in mind, our goal is to

1. define the matrix nearness problems that cover more general notions of the "stability",
2. develop appropriate tools for their solution,
3. construct efficient computational techniques for practical use.

Therefore, we are interested in:

- domains of the "stability" in its general setting,

The goal

Having this in mind, our goal is to

1. define the matrix nearness problems that cover more general notions of the "stability",
2. develop appropriate tools for their solution,
3. construct efficient computational techniques for practical use.

Therefore, we are interested in:

- domains of the "stability" in its general setting,
- behaviour of the eigenvalues under perturbations,

The goal

Having this in mind, our goal is to

1. define the matrix nearness problems that cover more general notions of the "stability",
2. develop appropriate tools for their solution,
3. construct efficient computational techniques for practical use.

Therefore, we are interested in:

- domains of the "stability" in its general setting,
- behaviour of the eigenvalues under perturbations,
- computational techniques for eigenvalue optimisation problems.

LYAPUNOV-TYPE EIGENVALUE LOCALIZATIONS

Lyapunov-type localization domains

Given a Hermitian matrix $\Gamma_f = \Gamma_f^* = [\gamma_{pq}] \in \mathbb{C}^{m,m}$, $m \geq 2$, and a set of the linearly independent holomorphic complex functions $\{\varphi_p\}_{p=1}^m$, we consider functions of the form

$$f(z) := \sum_{p=1}^{m} \sum_{q=1}^{m} \gamma_{pq} \varphi_p(z) \overline{\varphi_q(z)} = \varphi(z)^T \Gamma \, \overline{\varphi(z)},$$

where $\varphi(z) = [\varphi_1(z), \varphi_2(z), \ldots, \varphi_m(z)]^T$.

Lyapunov-type localization domains

Given a Hermitian matrix $\Gamma_f = \Gamma_f^* = [\gamma_{pq}] \in \mathbb{C}^{m,m}$, $m \geq 2$, and a set of the linearly independent holomorphic complex functions $\{\varphi_p\}_{p=1}^m$, we consider functions of the form

$$f(z) := \sum_{p=1}^{m} \sum_{q=1}^{m} \gamma_{pq} \varphi_p(z) \overline{\varphi_q(z)} = \varphi(z)^T \Gamma \, \overline{\varphi(z)},$$

where $\varphi(z) = [\varphi_1(z), \varphi_2(z), \ldots, \varphi_m(z)]^T$.

Since $\Gamma_f = \Gamma_f^*$, f is a real valued function of a complex variable and we can consider it as a map $f : \mathbb{R}^2 \to \mathbb{R}$, and define the domains in the complex plane:

$$\Lambda_f^+ := \{z \in \mathbb{C} : f(z) > 0\}$$
$$\Lambda_f^- := \{z \in \mathbb{C} : f(z) < 0\}$$
$$\Lambda_f^0 := \{z \in \mathbb{C} : f(z) = 0\}$$

Lyapunov-type localization domains

Given a Hermitian matrix $\Gamma_f = \Gamma_f^* = [\gamma_{pq}] \in \mathbb{C}^{m,m}$, $m \geq 2$, and a set of the linearly independent holomorphic complex functions $\{\varphi_p\}_{p=1}^m$, we consider functions of the form

$$f(z) := \sum_{p=1}^{m} \sum_{q=1}^{m} \gamma_{pq} \varphi_p(z) \overline{\varphi_q(z)} = \varphi(z)^T \Gamma \, \overline{\varphi(z)},$$

where $\varphi(z) = [\varphi_1(z), \varphi_2(z), \ldots, \varphi_m(z)]^T$.

Since $\Gamma_f = \Gamma_f^*$, f is a real valued function of a complex variable and we can consider it as a map $f : \mathbb{R}^2 \to \mathbb{R}$, and define the domains in the complex plane:

$$\begin{aligned} \Lambda_f^+ &:= \{z \in \mathbb{C} : f(z) > 0\} \\ \Lambda_f^- &:= \{z \in \mathbb{C} : f(z) < 0\} \\ \Lambda_f^0 &:= \{z \in \mathbb{C} : f(z) = 0\} \end{aligned}$$

For notational convenience, in the remainder we identify $f(z) = f(x + \imath y)$ with $f(x, y)$, i.e., $f(z) = f(x, y)$.

Lyapunov-type localization domains

Hermitian functions in standard basis:

Lyapunov-type localization domains

Hermitian functions in standard basis:

$$f(z) := \sum_{p=1}^{m} \sum_{q=1}^{m} \gamma_{pq} (z)^{p-1} (\overline{z})^{q-1} = \varphi(z)^T \Gamma_f \, \overline{\varphi(z)},$$

where $\Gamma_f = \Gamma_f^*$ and $\varphi(z) = [1 \; z \; z^2 \ldots z^{m-1}]^T$

Lyapunov-type localization domains

Hermitian functions in standard basis:

$$f(z) := \sum_{p=1}^{m} \sum_{q=1}^{m} \gamma_{pq} (z)^{p-1} (\overline{z})^{q-1} = \varphi(z)^T \Gamma_f \, \overline{\varphi(z)},$$

where $\Gamma_f = \Gamma_f^*$ and $\varphi(z) = [1 \; z \; z^2 \ldots z^{m-1}]^T$

The domains in the complex plane:

$$\Lambda_f^+ := \{z \in \mathbb{C} : f(z) > 0\}$$

$$\Lambda_f^- := \{z \in \mathbb{C} : f(z) < 0\}$$

$$\Lambda_f^0 := \{z \in \mathbb{C} : f(z) = 0\}$$

Lyapunov-type localization domains

Hermitian functions in standard basis:

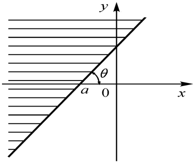
$$f(z) := \sum_{p=1}^{m}\sum_{q=1}^{m} \gamma_{pq}(z)^{p-1}(\overline{z})^{q-1} = \varphi(z)^T \Gamma_f \overline{\varphi(z)},$$

$$\text{where } \Gamma_f = \Gamma_f^* \text{ and } \varphi(z) = [1\ z\ z^2 \dots z^{m-1}]^T$$

The domains in the complex plane:

$$\Lambda_f^+ := \{z \in \mathbb{C} : f(z) > 0\}$$
$$\Lambda_f^- := \{z \in \mathbb{C} : f(z) < 0\}$$
$$\Lambda_f^0 := \{z \in \mathbb{C} : f(z) = 0\}$$

Straight line $y\cos\theta = (x-a)\sin\theta$:



$$\Gamma_f = \begin{bmatrix} 2a\sin\theta & -\sin\theta + \imath\cos\theta \\ -\sin\theta - \imath\cos\theta & 0 \end{bmatrix}$$

Lyapunov-type localization domains

Hermitian functions in standard basis:

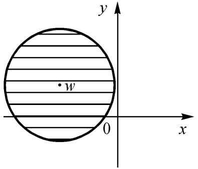
$$f(z) := \sum_{p=1}^{m}\sum_{q=1}^{m} \gamma_{pq}(z)^{p-1}(\overline{z})^{q-1} = \varphi(z)^T \Gamma_f \, \overline{\varphi(z)},$$

where $\Gamma_f = \Gamma_f^*$ and $\varphi(z) = [1 \; z \; z^2 \dots z^{m-1}]^T$

The domains in the complex plane:

$$\Lambda_f^+ := \{z \in \mathbb{C} : f(z) > 0\}$$
$$\Lambda_f^- := \{z \in \mathbb{C} : f(z) < 0\}$$
$$\Lambda_f^0 := \{z \in \mathbb{C} : f(z) = 0\}$$

Circle $(x-a)^2 + (y-b)^2 = r^2, \; w = a + \imath b$:



$$\Gamma_f = \begin{bmatrix} r^2 - |w|^2 & w \\ \overline{w} & -1 \end{bmatrix}$$

Lyapunov-type localization domains

Hermitian functions in standard basis:

$$f(z) := \sum_{p=1}^{m}\sum_{q=1}^{m}\gamma_{pq}(z)^{p-1}(\overline{z})^{q-1} = \varphi(z)^T\Gamma_f\,\overline{\varphi(z)},$$

where $\Gamma_f = \Gamma_f^*$ and $\varphi(z) = [1\ z\ z^2 \ldots z^{m-1}]^T$

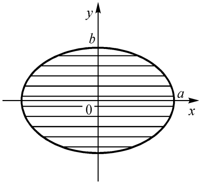
The domains in the complex plane:

$$\Lambda_f^+ := \{z \in \mathbb{C} : f(z) > 0\}$$

$$\Lambda_f^- := \{z \in \mathbb{C} : f(z) < 0\}$$

$$\Lambda_f^0 := \{z \in \mathbb{C} : f(z) = 0\}$$

Elipse $x^2/a^2 + y^2/b^2 = 1,\ a > 0, b > 0$:



$$\Gamma_f = \begin{bmatrix} 4a^2b^2 & 0 & a^2-b^2 \\ 0 & -2(a^2+b^2) & 0 \\ a^2-b^2 & 0 & 0 \end{bmatrix}$$

Lyapunov-type localization domains

Hermitian functions in standard basis:

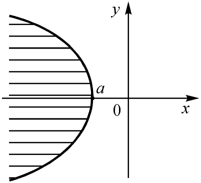
$$f(z) := \sum_{p=1}^{m}\sum_{q=1}^{m} \gamma_{pq}(z)^{p-1}(\overline{z})^{q-1} = \varphi(z)^T \Gamma_f \, \overline{\varphi(z)},$$

where $\Gamma_f = \Gamma_f^*$ and $\varphi(z) = [1 \; z \; z^2 \ldots z^{m-1}]^T$

The domains in the complex plane:

$$\Lambda_f^+ := \{z \in \mathbb{C} : f(z) > 0\}$$
$$\Lambda_f^- := \{z \in \mathbb{C} : f(z) < 0\}$$
$$\Lambda_f^0 := \{z \in \mathbb{C} : f(z) = 0\}$$

Parabola $x = a - by^2, \; a < 0, b > 0$:



$$\Gamma_f = \begin{bmatrix} 2a & -1 & b/2 \\ -1 & -b & 0 \\ b/2 & 0 & 0 \end{bmatrix}$$

Lyapunov-type localization domains

Hermitian functions in standard basis:

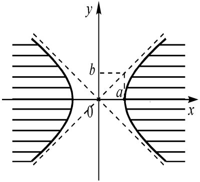
$$f(z) := \sum_{p=1}^{m}\sum_{q=1}^{m} \gamma_{pq}(z)^{p-1}(\overline{z})^{q-1} = \varphi(z)^T \Gamma_f \, \overline{\varphi(z)},$$

where $\Gamma_f = \Gamma_f^*$ and $\varphi(z) = [1\ z\ z^2 \ldots z^{m-1}]^T$

The domains in the complex plane:

$$\Lambda_f^+ := \{z \in \mathbb{C} : f(z) > 0\}$$
$$\Lambda_f^- := \{z \in \mathbb{C} : f(z) < 0\}$$
$$\Lambda_f^0 := \{z \in \mathbb{C} : f(z) = 0\}$$

Hyperbola $x^2/a^2 - y^2/b^2 = 1,\ 0 < b \leq a$:



$$\Gamma_f = \begin{bmatrix} -4a^2b^2 & 0 & a^2+b^2 \\ 0 & -2(b^2-a^2) & 0 \\ a^2+b^2 & 0 & 0 \end{bmatrix}$$

Lyapunov-type localization domains

Hermitian functions in standard basis:

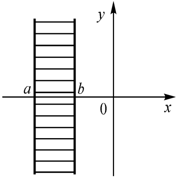
$$f(z) := \sum_{p=1}^{m} \sum_{q=1}^{m} \gamma_{pq}(z)^{p-1}(\overline{z})^{q-1} = \varphi(z)^T \Gamma_f \overline{\varphi(z)},$$

where $\Gamma_f = \Gamma_f^*$ and $\varphi(z) = [1\ z\ z^2 \ldots z^{m-1}]^T$

The domains in the complex plane:

$$\Lambda_f^+ := \{z \in \mathbb{C} : f(z) > 0\}$$
$$\Lambda_f^- := \{z \in \mathbb{C} : f(z) < 0\}$$
$$\Lambda_f^0 := \{z \in \mathbb{C} : f(z) = 0\}$$

Vertical strip $(x-a)(b-x) = 0,\ a < b$:



$$\Gamma_f = \begin{bmatrix} -2a & a+b & -1/2 \\ a+b & -1 & 0 \\ -1/2 & 0 & 0 \end{bmatrix}$$

Lyapunov-type localization domains

Hermitian functions in standard basis:

$$f(z) := \sum_{p=1}^{m} \sum_{q=1}^{m} \gamma_{pq} (z)^{p-1} (\overline{z})^{q-1} = \varphi(z)^T \Gamma_f \, \overline{\varphi(z)},$$

where $\Gamma_f = \Gamma_f^*$ and $\varphi(z) = [1 \; z \; z^2 \ldots z^{m-1}]^T$

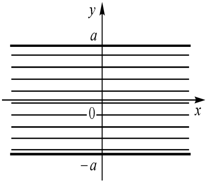
The domains in the complex plane:

$$\Lambda_f^+ := \{z \in \mathbb{C} : f(z) > 0\}$$

$$\Lambda_f^- := \{z \in \mathbb{C} : f(z) < 0\}$$

$$\Lambda_f^0 := \{z \in \mathbb{C} : f(z) = 0\}$$

Horizontal strip $y^2 = a^2, \; a > 0$:



$$\Gamma_f = \begin{bmatrix} 4a^2 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

Lyapunov-type localization domains

Hermitian functions in standard basis:

$$f(z) := \sum_{p=1}^{m} \sum_{q=1}^{m} \gamma_{pq} (z)^{p-1} (\bar{z})^{q-1} = \varphi(z)^T \Gamma_f \, \overline{\varphi(z)},$$

where $\Gamma_f = \Gamma_f^*$ and $\varphi(z) = [1 \; z \; z^2 \ldots z^{m-1}]^T$

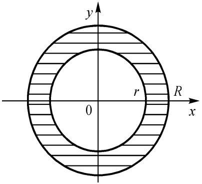
The domains in the complex plane:

$$\Lambda_f^+ := \{z \in \mathbb{C} : f(z) > 0\}$$

$$\Lambda_f^- := \{z \in \mathbb{C} : f(z) < 0\}$$

$$\Lambda_f^0 := \{z \in \mathbb{C} : f(z) = 0\}$$

Two circles $(x^2 + y^2 - r^2)(R^2 - x^2 - y^2) = 0, \; 0 < r < R$:



$$\Gamma_f = \begin{bmatrix} -r^2 R^2 & 0 & 0 \\ 0 & r^2 + R^2 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

The Generalized Lyapunov Theorem

Theorem (Mazko 2008)

Given a Hermitian function f defined by the Hermitian matrix Γ_f, such that the matrix $\Gamma_f \varphi(\overline{z})\varphi(z)^T \Gamma_f - f(z)\Gamma_f$ is Hermitian positive semidefinite, define the operator

$$\mathcal{L}_A^f(X) := \sum_{p=1}^{m}\sum_{q=1}^{m} \gamma_{pq} A^{p-1} X A^{*(q-1)}.$$

Then, for an arbitrary matrix $A \in \mathbb{C}^{n,n}$ and an arbitrary Hermitian positive definite matrix $Y \in \mathbb{C}^{n,n}$, all the eigenvalues of matrix A belong to the domain Λ_f^+ if and only if the equation $\mathcal{L}_A^f(X) = Y$ has a unique positive definite solution X, i.e.,

$$\Lambda(A) \subseteq \Lambda_f^+ \text{ if and only if } \mathcal{L}_A^f : \mathbb{H}^{n,n} \to \mathbb{H}^{n,n} \text{ is a bijection.}$$

A.G. Mazko: *Matrix Equations, Spectral Problems and Stability of Dynamic Systems*. Cambridge Scientific Publishers Ltd, 2008

THE DISTANCE TO DELOCALIZATION
&
THE DISTANCE TO LOCALIZATION

Distance to delocalization/localization

Given arbitrary $\Gamma_f = \Gamma_f^*$, let Λ_f^+, Λ_f^0 and Λ_f^- be the domains where $f(z) = \varphi(\overline{z})^* \Gamma_f \varphi(z)$ is positive, zero and negative, respectively

Distance to delocalization/localization

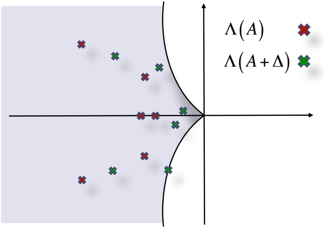
Given arbitrary $\Gamma_f = \Gamma_f^*$, let Λ_f^+, Λ_f^0 and Λ_f^- be the domains where $f(z) = \varphi(\bar{z})^* \Gamma_f \varphi(z)$ is positive, zero and negative, respectively

Distance to delocalization:

For a given A such that $\Lambda(A) \subseteq \Lambda_f^+$ solve

$$\delta_f^-(A) := \sup \varepsilon \quad \text{s.t. } \Lambda_\varepsilon(A) \subseteq \Lambda_f^+$$

(K.,M. & S., SIMAX 2015)



Distance to delocalization/localization

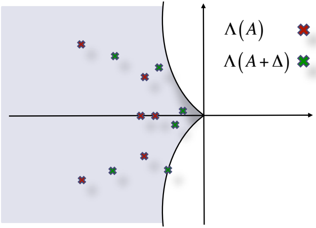
Given arbitrary $\Gamma_f = \Gamma_f^*$, let Λ_f^+, Λ_f^0 and Λ_f^- be the domains where $f(z) = \varphi(\overline{z})^* \Gamma_f \varphi(z)$ is positive, zero and negative, respectively

Distance to delocalization:

For a given A such that $\Lambda(A) \subseteq \Lambda_f^+$ solve

$$\delta_f^-(A) := \sup \varepsilon \quad \text{s.t. } \Lambda_\varepsilon(A) \subseteq \Lambda_f^+$$

(K.,M. & S., SIMAX 2015)

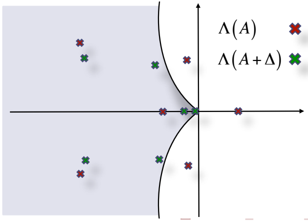


Distance to localization:

For a given A such that $\Lambda(A) \not\subseteq \Lambda_f^+$ solve

$$\delta_f^+(A) := \inf_{X,Y} \|X - A\| \quad \text{s.t. } \mathcal{L}_X^f(Y) \succ 0, \quad Y \succ 0$$

(MFO RIP project, in preparation)

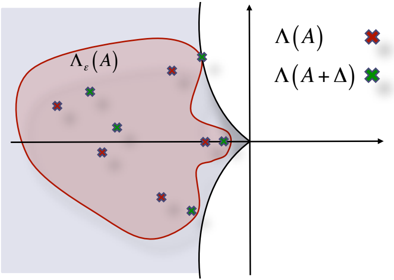


COMPUTATIONAL METHODS FOR THE DISTANCE TO DELOCALIZATION

Problem formulation and the assumptions

For a given A such that $\Lambda(A) \subseteq \Lambda_f^+$ solve

$$\delta_f^-(A) = \sup \varepsilon \quad \text{s.t. } \Lambda_\varepsilon(A) \subseteq \Lambda_f^+$$

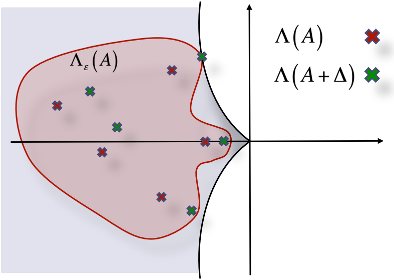


Problem formulation and the assumptions

For a given A such that $\Lambda(A) \subseteq \Lambda_f^+$ solve

$$\delta_f^-(A) = \sup \varepsilon \quad \text{s.t.} \quad \Lambda_\varepsilon(A) \subseteq \Lambda_f^+$$

$$\widehat{\varepsilon} = \min_{z \in \Lambda_f^0} \sigma_{min}(A - zI)$$

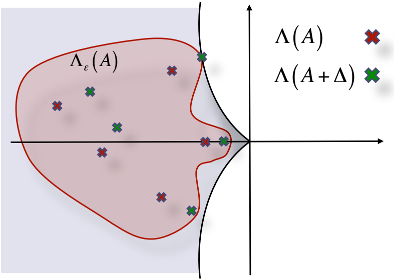


Problem formulation and the assumptions

For a given A such that $\Lambda(A) \subseteq \Lambda_f^+$ solve

$$\delta_f^-(A) = \sup \varepsilon \quad \text{s.t. } \Lambda_\varepsilon(A) \subseteq \Lambda_f^+$$

$$\widehat{\varepsilon} = \min_{z \in \Lambda_f^0} \sigma_{min}(A - zI)$$

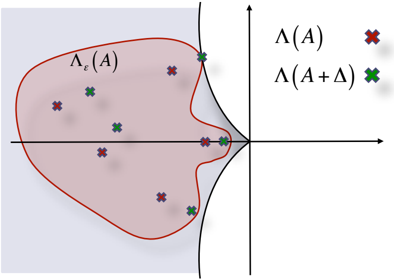


Problem formulation and the assumptions

For a given A such that $\Lambda(A) \subseteq \Lambda_f^+$ solve

$$\delta_f^-(A) = \sup \varepsilon \quad \text{s.t. } \Lambda_\varepsilon(A) \subseteq \Lambda_f^+$$

$$\widehat{\varepsilon} = \min_{z \in \Lambda_f^0} \sigma_{min}(A - zI)$$



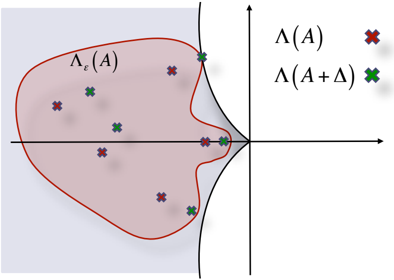
Assumptions:

Problem formulation and the assumptions

For a given A such that $\Lambda(A) \subseteq \Lambda_f^+$ solve

$$\delta_f^-(A) = \sup \varepsilon \quad \text{s.t. } \Lambda_\varepsilon(A) \subseteq \Lambda_f^+$$

$$\widehat{\varepsilon} = \min_{z \in \Lambda_f^0} \sigma_{min}(A - zI)$$



Assumptions:

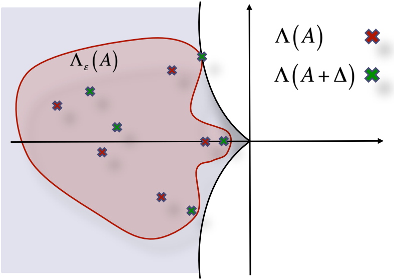
1. For the domains defined by $f(z)$, $\Lambda(A) \subseteq \Lambda_f^+$ and $\partial\Lambda_f^+ = \partial\Lambda_f^- = \Lambda_f^0$.

Problem formulation and the assumptions

For a given A such that $\Lambda(A) \subseteq \Lambda_f^+$ solve

$$\delta_f^-(A) = \sup \varepsilon \quad \text{s.t. } \Lambda_\varepsilon(A) \subseteq \Lambda_f^+$$

$$\widehat{\varepsilon} = \min_{z \in \Lambda_f^0} \sigma_{min}(A - zI)$$



Assumptions:

1. For the domains defined by $f(z)$, $\Lambda(A) \subseteq \Lambda_f^+$ and $\partial\Lambda_f^+ = \partial\Lambda_f^- = \Lambda_f^0$.
2. The distance to delocalization $\delta_f^-(A)$ is achieved at a simple singular value of $A - \widehat{z}I$, where $\widehat{z}$ is point where the solution of the above problem is achieved.

Overview of the computational issues:

Under the assumptions, we have developed

Overview of the computational issues:

Under the assumptions, we have developed

1. local explicit distance to delocalization algorithm (eD2D) mainly intended for medium size problems,

Overview of the computational issues:

Under the assumptions, we have developed

1. local explicit distance to delocalization algorithm (eD2D) mainly intended for medium size problems,

2. local implicit distance to delocalization algorithm (iD2D) mainly intended for large (sparse) problems,

Overview of the computational issues:

Under the assumptions, we have developed

1. local explicit distance to delocalization algorithm (eD2D) mainly intended for medium size problems,
2. local implicit distance to delocalization algorithm (iD2D) mainly intended for large (sparse) problems,
3. control of the iteration step size (damping) to enhance the stability of the iteration sequences - $\mathrm{eD2D}_d$ and $\mathrm{iD2D}_d$,,

Overview of the computational issues:

Under the assumptions, we have developed

1. local explicit distance to delocalization algorithm (eD2D) mainly intended for medium size problems,
2. local implicit distance to delocalization algorithm (iD2D) mainly intended for large (sparse) problems,
3. control of the iteration step size (damping) to enhance the stability of the iteration sequences - eD2D_d and iD2D_d,,
4. globality test (piece-wise linear and/or circular domain boundary) as an outer iteration to produce global methods - eD2D(g) and iD2D(g),

Overview of the computational issues:

Under the assumptions, we have developed

1. local explicit distance to delocalization algorithm (eD2D) mainly intended for medium size problems,
2. local implicit distance to delocalization algorithm (iD2D) mainly intended for large (sparse) problems,
3. control of the iteration step size (damping) to enhance the stability of the iteration sequences - eD2D_d and iD2D_d,,
4. globality test (piece-wise linear and/or circular domain boundary) as an outer iteration to produce global methods - eD2D(g) and iD2D(g),
5. strategies for choosing appropriate starting points for D2D algorithms.

Explicit distance to delocalization algorithm

If we define $s(x,y) := \sigma_{min}(A - (x + \imath y)I)$, our aim is to determine $(\widehat{x}, \widehat{y}) \in \mathbb{R}^2$ such that

$$s(\widehat{x}, \widehat{y}) = \min\{s(x,y) \,:\, f(x,y) = 0,\, x, y \in \mathbb{R}\}.$$

Explicit distance to delocalization algorithm

If we define $s(x, y) := \sigma_{min}(A - (x + \imath y)I)$, our aim is to determine $(\widehat{x}, \widehat{y}) \in \mathbb{R}^2$ such that

$$s(\widehat{x}, \widehat{y}) = \min\{s(x, y) \,:\, f(x, y) = 0,\, x, y \in \mathbb{R}\}.$$

In the neighborhood of $(\widehat{x}, \widehat{y})$ the following equalities hold

$$\begin{aligned}
s_x(x, y) &= -\mathrm{Re}(u^* v),\\
s_y(x, y) &= \mathrm{Im}(u^* v),\\
s_{xx}(x, y) &= \varepsilon u^* E u + \varepsilon v^* F v + 2\mathrm{Re}(v^*(A - zI)Eu) + \varepsilon^{-1}(\mathrm{Im}(u^* v))^2,\\
s_{xy}(x, y) &= 2\mathrm{Im}(v^*(A - zI)Eu) + \varepsilon^{-1}\mathrm{Re}(u^* v)\mathrm{Im}(u^* v),\\
s_{yy}(x, y) &= \varepsilon u^* E u + \varepsilon v^* F v - 2\mathrm{Re}(v^*(A - zI)Eu) + \varepsilon^{-1}(\mathrm{Re}(u^* v))^2.
\end{aligned}$$

Here,

$$E = (\varepsilon^2 I - (A - zI)^*(A - zI))^\dagger \quad \text{and} \quad F = (\varepsilon^2 I - (A - zI)(A - zI)^*)^\dagger,$$

where $\dagger$ denotes the Moore-Penrose pseudoinverse, and (ε, u, v) is the minimal singular triplet of $A - zI$ with $z = x + \imath y$.

Explicit distance to delocalization algorithm

If we define $s(x,y) := \sigma_{min}(A-(x+\imath y)I)$, our aim is to determine $(\widehat{x},\widehat{y}) \in \mathbb{R}^2$ such that

$$s(\widehat{x},\widehat{y}) = \min\{s(x,y)\ :\ f(x,y)=0,\ x,y\in\mathbb{R}\}.$$

While, the derivatives of the Hermitian function $f(x,y)$ in the standard basis can be expressed as:

$$\begin{aligned}
f_x(x,y) &= \varphi^T(x,y)\,[D^*\Gamma_f+\Gamma_f D]\,\varphi(x,-y),\\
f_y(x,y) &= \imath\,\varphi^T(x,y)\,[D^*\Gamma_f-\Gamma_f D]\,\varphi(x,-y),\\
f_{xx}(x,y) &= \varphi^T(x,y)\left[D^{2*}\Gamma_f+2D^*\Gamma_f D+\Gamma_f D^2\right]\varphi(x,-y),\\
f_{xy}(x,y) &= \imath\,\varphi^T(x,y)\left[D^{2*}\Gamma_f-\Gamma_f D^2\right]\varphi(x,-y),\\
f_{yy}(x,y) &= -\varphi^T(x,y)\left[D^{2*}\Gamma_f-2D^*\Gamma_f D+\Gamma_f D^2\right]\varphi(x,-y),
\end{aligned}$$

where

$$D = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ & & & m-1 & 0 \end{bmatrix} \qquad \varphi(x,y) = \begin{bmatrix} 1 \\ x+\imath y \\ (x+\imath y)^2 \\ \vdots \\ (x+\imath y)^{m-1} \end{bmatrix}$$

Explicit distance to delocalization algorithm

If we define $s(x,y) := \sigma_{min}(A-(x+\imath y)I)$, our aim is to determine $(\widehat{x},\widehat{y}) \in \mathbb{R}^2$ such that

$$s(\widehat{x},\widehat{y}) = \min\{s(x,y)\ :\ f(x,y)=0,\ x,y\in\mathbb{R}\}.$$

While, the derivatives of the Hermitian function $f(x,y)$ in the standard basis can be expressed as:

$$\begin{aligned}
f_x(x,y) &= \varphi^T(x,y)\,[D^*\Gamma_f + \Gamma_f D]\,\varphi(x,-y),\\
f_y(x,y) &= \imath\,\varphi^T(x,y)\,[D^*\Gamma_f - \Gamma_f D]\,\varphi(x,-y),\\
f_{xx}(x,y) &= \varphi^T(x,y)\,\left[D^{2*}\Gamma_f + 2D^*\Gamma_f D + \Gamma_f D^2\right]\varphi(x,-y),\\
f_{xy}(x,y) &= \imath\,\varphi^T(x,y)\,\left[D^{2*}\Gamma_f - \Gamma_f D^2\right]\varphi(x,-y),\\
f_{yy}(x,y) &= -\varphi^T(x,y)\,\left[D^{2*}\Gamma_f - 2D^*\Gamma_f D + \Gamma_f D^2\right]\varphi(x,-y),
\end{aligned}$$

where

$$D = \begin{bmatrix} 0 & & & & \\ 1 & 0 & & & \\ & 2 & 0 & & \\ & & \ddots & \ddots & \\ & & & m-1 & 0 \end{bmatrix} \qquad \varphi(x,y) = \begin{bmatrix} 1 \\ x+\imath y \\ (x+\imath y)^2 \\ \vdots \\ (x+\imath y)^{m-1} \end{bmatrix}$$

Just $m \times m$ matrix-vector multiplications!

Explicit distance to delocalization algorithm

Thus, we can introduce the Lagrange function

$$\Phi(x, y, \mu) := s(x, y) + \mu f(x, y),$$

where μ is a Lagrange multiplier, and solve minimisation problem applying Newton's method given by

$$\xi^{(k+1)} = \xi^{(k)} - \left[\nabla^2 \Phi\left(\xi^{(k)}\right)\right]^{-1} \nabla \Phi\left(\xi^{(k)}\right), \quad k = 0, 1, 2, \ldots,$$

where $\xi = [x, y, \mu]^T$ and

$$\nabla \Phi = \begin{bmatrix} s_x + \mu f_x \\ s_y + \mu f_y \\ f \end{bmatrix}, \qquad \nabla^2 \Phi = \begin{bmatrix} s_{xx} + \mu f_{xx} & s_{xy} + \mu f_{xy} & f_x \\ s_{xy} + \mu f_{xy} & s_{yy} + \mu f_{yy} & f_y \\ f_x & f_y & 0 \end{bmatrix}.$$

For the sake of brevity, here we omit the arguments (x, y, μ).

THE IMPLICIT DISTANCE TO DELOCALIZATION ALGORITHM

Implicit determinant approach...

We use the ideas from the papers

A. Spence and C. Poulton.: *Photonic band structure calculations using nonlinear eigenvalue techniques*. J. Comput. Phys., 204(1):65–81, 2005.

M. A. Freitag and A. Spence.: *A Newton-based method for the calculation of the distance to instability.*. Linear Algebra Appl., 435(12):3189–3205, 2011.

on the implicit determinant method to replace intensive SVD computations by LU factorizations, which significantly reduces the overall computational cost.

Implicit determinant approach...

We use the ideas from the papers

A. Spence and C. Poulton.: *Photonic band structure calculations using nonlinear eigenvalue techniques*. J. Comput. Phys., 204(1):65–81, 2005.

M. A. Freitag and A. Spence.: *A Newton-based method for the calculation of the distance to instability.*. Linear Algebra Appl., 435(12):3189–3205, 2011.

on the implicit determinant method to replace intensive SVD computations by LU factorizations, which significantly reduces the overall computational cost.

To that end, we start with the following formulation of the D2D problem:

$$\min\ \varepsilon > 0 \quad \text{s.t.} \quad \begin{array}{l} (A-(x+\imath y)I)v = \varepsilon u, \\ (A^*-(x-\imath y)I)u = \varepsilon v, \\ f(x,y) = 0, \end{array} \qquad u,v\in\mathbb{C}^n,\ x,y,\varepsilon\in\mathbb{R}.$$

Implicit distance to delocalization algorithm

For a given $z = x + \imath y \in \mathbb{C}$ and $\varepsilon > 0$ define:

$$H(x,y,\varepsilon) = \begin{bmatrix} -\varepsilon I & A - (x+\imath y)I \\ A^* - (x - \imath y)I & -\varepsilon I \end{bmatrix} \text{ and}$$

$$M(x,y,\varepsilon) = \begin{bmatrix} H(x,y,\varepsilon) & c \\ c^* & 0 \end{bmatrix}$$

Implicit distance to delocalization algorithm

For a given $z = x + \imath y \in \mathbb{C}$ and $\varepsilon > 0$ define:

$$H(x,y,\varepsilon) = \begin{bmatrix} -\varepsilon I & A-(x+\imath y)I \\ A^*-(x-\imath y)I & -\varepsilon I \end{bmatrix} \text{ and}$$

$$M(x,y,\varepsilon) = \begin{bmatrix} H(x,y,\varepsilon) & c \\ c^* & 0 \end{bmatrix}$$

and observe linear system

$$\begin{bmatrix} H(x,y,\varepsilon) & c \\ c^* & 0 \end{bmatrix} \begin{bmatrix} g(x,y,\varepsilon) \\ h(x,y,\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Implicit distance to delocalization algorithm

For a given $z = x + \imath y \in \mathbb{C}$ and $\varepsilon > 0$ define:

$$H(x,y,\varepsilon) = \begin{bmatrix} -\varepsilon I & A - (x+\imath y)I \\ A^* - (x-\imath y)I & -\varepsilon I \end{bmatrix} \text{ and}$$

$$M(x,y,\varepsilon) = \begin{bmatrix} H(x,y,\varepsilon) & c \\ c^* & 0 \end{bmatrix}$$

and observe linear system

$$\begin{bmatrix} H(x,y,\varepsilon) & c \\ c^* & 0 \end{bmatrix} \begin{bmatrix} g(x,y,\varepsilon) \\ h(x,y,\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Having that $M(x,y,\varepsilon)$ is nonsingular, $h(x,y,\varepsilon) = \frac{\det H(x,y,\varepsilon)}{\det M(x,y,\varepsilon)}$ can be computed by LU factorization

Implicit distance to delocalization algorithm

For a given $z = x + \imath y \in \mathbb{C}$ and $\varepsilon > 0$ define:

$$H(x,y,\varepsilon) = \begin{bmatrix} -\varepsilon I & A - (x+\imath y)I \\ A^* - (x-\imath y)I & -\varepsilon I \end{bmatrix} \text{ and}$$

$$M(x,y,\varepsilon) = \begin{bmatrix} H(x,y,\varepsilon) & c \\ c^* & 0 \end{bmatrix}$$

and observe linear system

$$\begin{bmatrix} H(x,y,\varepsilon) & c \\ c^* & 0 \end{bmatrix} \begin{bmatrix} g(x,y,\varepsilon) \\ h(x,y,\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Having that $M(x,y,\varepsilon)$ is nonsingular, $h(x,y,\varepsilon) = \frac{\det H(x,y,\varepsilon)}{\det M(x,y,\varepsilon)}$ can be computed by LU factorization
- $h(x,y,\varepsilon) = 0$ if and only if ε is a singular value of A

Implicit distance to delocalization algorithm

For a given $z = x + \imath y \in \mathbb{C}$ and $\varepsilon > 0$ define:

$$H(x, y, \varepsilon) = \begin{bmatrix} -\varepsilon I & A - (x + \imath y)I \\ A^* - (x - \imath y)I & -\varepsilon I \end{bmatrix} \text{ and}$$

$$M(x, y, \varepsilon) = \begin{bmatrix} H(x, y, \varepsilon) & c \\ c^* & 0 \end{bmatrix}$$

and observe linear system

$$\begin{bmatrix} H(x, y, \varepsilon) & c \\ c^* & 0 \end{bmatrix} \begin{bmatrix} g(x, y, \varepsilon) \\ h(x, y, \varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

- Having that $M(x, y, \varepsilon)$ is nonsingular, $h(x, y, \varepsilon) = \frac{\det H(x,y,\varepsilon)}{\det M(x,y,\varepsilon)}$ can be computed by LU factorization
- $h(x, y, \varepsilon) = 0$ if and only if ε is a singular value of A
- $\partial\Lambda_\varepsilon(A)$ is the outermost closed curve of $\{(x, y) \in \mathbb{R}^2 : h(x, y, \varepsilon) = 0\}$

Implicit distance to delocalization algorithm

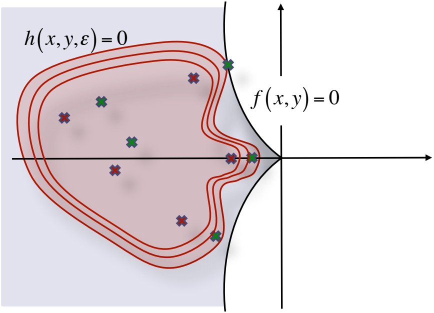
For given hermitian matrix Γ_f and matrix A, such that $\Lambda(A) \subseteq \Lambda_f^+$, solve:

$$\begin{aligned} \min \quad & \varepsilon^2 \\ \text{s.t.} \quad & h(x, y, \varepsilon) = 0 \\ & f(x, y) = 0 \\ & x, y, \varepsilon \in \mathbb{R} \end{aligned}$$

Implicit distance to delocalization algorithm

For given hermitian matrix Γ_f and matrix A, such that $\Lambda(A) \subseteq \Lambda_f^+$, solve:

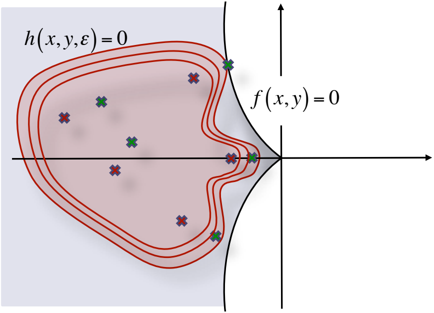
$$\begin{aligned} \min \quad & \varepsilon^2 \\ \text{s.t.} \quad & h(x,y,\varepsilon) = 0 \\ & f(x,y) = 0 \\ & x, y, \varepsilon \in \mathbb{R} \end{aligned}$$



Implicit distance to delocalization algorithm

For given hermitian matrix Γ_f and matrix A, such that $\Lambda(A) \subseteq \Lambda_f^+$, solve:

$$\begin{aligned} \min \quad & \varepsilon^2 \\ \text{s.t.} \quad & h(x,y,\varepsilon) = 0 \\ & f(x,y) = 0 \\ & x, y, \varepsilon \in \mathbb{R} \end{aligned}$$



$$\min \quad \Psi(x,y,\varepsilon,\lambda,\mu),$$

$$\Psi(x,y,\varepsilon,\lambda,\mu) := \varepsilon^2 + \lambda h(x,y,\varepsilon) + \mu f(x,y)$$

Newton's method for solving $\nabla\Psi = 0$...

Implicit distance to delocalization algorithm

Derivatives of $h(x, y, \varepsilon)$:

$$\underbrace{\begin{bmatrix} -\varepsilon I & A - (x + \imath y) I & c_1 \\ A^* - (x - \imath y) I & -\varepsilon I & c_2 \\ c_1^* & c_2^* & 0 \end{bmatrix}}_{M(x,y,\varepsilon)} \begin{bmatrix} u(x, y, \varepsilon) \\ v(x, y, \varepsilon) \\ h(x, y, \varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Implicit distance to delocalization algorithm

Derivatives of $h(x,y,\varepsilon)$:

$$\underbrace{\begin{bmatrix} -\varepsilon I & A-(x+\imath y)I & c_1 \\ A^*-(x-\imath y)I & -\varepsilon I & c_2 \\ c_1^* & c_2^* & 0 \end{bmatrix}}_{M(x,y,\varepsilon)} \begin{bmatrix} u(x,y,\varepsilon) \\ v(x,y,\varepsilon) \\ h(x,y,\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} M \end{bmatrix} \left[\begin{array}{ccc|cccccc} u_x & u_y & u_\varepsilon & u_{xx} & u_{xy} & u_{yy} & u_{x\varepsilon} & u_{y\varepsilon} & u_{\varepsilon\varepsilon} \\ v_x & v_y & v_\varepsilon & v_{xx} & v_{xy} & v_{yy} & v_{x\varepsilon} & v_{y\varepsilon} & v_{\varepsilon\varepsilon} \\ h_x & h_y & h_\varepsilon & h_{xx} & h_{xy} & h_{yy} & h_{x\varepsilon} & h_{y\varepsilon} & h_{\varepsilon\varepsilon} \end{array}\right]$$

$$= \left[\begin{array}{ccc|cccccc} v & \imath v & u & 2v_x & v_y+\imath v_x & 2\imath v_y & u_x+v_\varepsilon & u_y+\imath v_\varepsilon & 2u_\varepsilon \\ u & -\imath u & v & 2u_x & u_y-\imath u_x & -2\imath u_y & v_x+u_\varepsilon & v_y-\imath u_\varepsilon & 2v_\varepsilon \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Implicit distance to delocalization algorithm

Derivatives of $h(x,y,\varepsilon)$:

$$\underbrace{\begin{bmatrix} -\varepsilon I & A-(x+\imath y)I & c_1 \\ A^*-(x-\imath y)I & -\varepsilon I & c_2 \\ c_1^* & c_2^* & 0 \end{bmatrix}}_{M(x,y,\varepsilon)} \begin{bmatrix} u(x,y,\varepsilon) \\ v(x,y,\varepsilon) \\ h(x,y,\varepsilon) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} M \end{bmatrix} \left[\begin{array}{ccc|cccccc} u_x & u_y & u_\varepsilon & u_{xx} & u_{xy} & u_{yy} & u_{x\varepsilon} & u_{y\varepsilon} & u_{\varepsilon\varepsilon} \\ v_x & v_y & v_\varepsilon & v_{xx} & v_{xy} & v_{yy} & v_{x\varepsilon} & v_{y\varepsilon} & v_{\varepsilon\varepsilon} \\ h_x & h_y & h_\varepsilon & h_{xx} & h_{xy} & h_{yy} & h_{x\varepsilon} & h_{y\varepsilon} & h_{\varepsilon\varepsilon} \end{array}\right]$$

$$= \left[\begin{array}{ccc|cccccc} v & \imath v & u & 2v_x & v_y+\imath v_x & 2\imath v_y & u_x+v_\varepsilon & u_y+\imath v_\varepsilon & 2u_\varepsilon \\ u & -\imath u & v & 2u_x & u_y-\imath u_x & -2\imath u_y & v_x+u_\varepsilon & v_y-\imath u_\varepsilon & 2v_\varepsilon \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

One LU factorization of $(2n+1)\times(2n+1)$ matrix $M(x,y,\varepsilon)$!

NUMERICAL EXAMPLES

Orr-Sommerfeld matrix & distance to c-instability

Let A be a matrix of size $n = 2000$ that originates from the Orr-Sommerfeld equation of parallel fluid flow in an idealized infinitely long domain with Reynolds number $R = 1000$ (EigTool function `orrsommerfeld_demo.m`)

i	$y^{(i)} = -0.3\dots$		$\varepsilon^{(i)} = 0.002\dots$		$\mathrm{Error}^{(i)}$	
	FS	iD2D	FS	iD2D	FS	iD2D
1	46284817581	46284817581	904423522	904423522	—	—
2	51841997634	51741589158	904423522	904423522	5.5574e-03	5.4573e-03
3	52041616428	52040927685	851016862	851945690	2.0119e-04	3.0033e-04
4	52043134040	52043133995	876134970	876101856	1.5177e-06	2.2069e-06
5	52043134062	52043134065	876154103	876154101	2.1732e-11	6.9504e-11
6	52043134066	52043134064	876154104	876154104	4.1219e-12	6.0692e-13
7	52043134068	—	876154104	—	1.8439e-12	—
8	52043134068	—	876154104	—	2.6605e-13	—

Comparing the number of inner iterations and the CPU time:

eD2D: 28 inner iterations in 21.55min

iD2D: 6 inner iterations in 1.86min

FS: 8 inner iterations in 3.52min

Tolosa matrix & distance to c-instability

Let A be a Tolosa matrix of size $n = 340$ from the Matrix Market repository (highly nonnormal, medium size and sparse, used in the stability analysis of a flying airplane).

i	$y^{(i)} = 155.999\ldots$		
	FS	eD2D	iD2D
1	9219999	9219999	9219999
2	8439555	8439945	8440335
3	8439945	8439945	8439945
4	8439945	8439945	8439945

i	$\varepsilon^{(i)} = 0.00\ldots$			Error$^{(i)}$		
	FS	eD2D	iD2D	FS	eD2D	iD2D
1	2001797137	2001797137	2001797137	—	—	—
2	2001797137	1999796887	2001797137	7.8070e-05	8.0226e-10	7.8018e-05
3	1999796637	1999796887	1999796638	3.8968e-08	1.3095e-14	3.9027e-08
4	1999796887	—	1999796887	2.5160e-14	—	8.2767e-15

Comparing the number of inner iterations and the CPU time:

eD2D: 3 inner iterations in 0.97sec

iD2D: 4 inner iterations in 0.58sec

FS: 4 inner iterations in 0.66sec

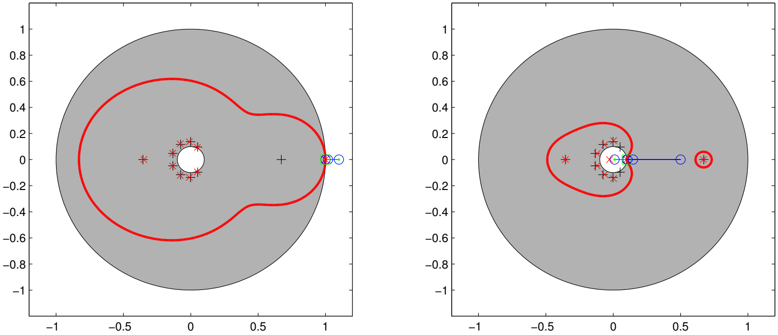
Leslie matrix & D2D for the annulus domain

Let $A = [a_{ij}]$ be a Leslie matrix describing the population of 10 age groups that has a geometric progression of birthrates and harmonic transition probabilities. To be more realistic, the fertility of the first age group is set to zero. More precisely,

$$a_{ij} = \begin{cases} aq^j, & \text{for } i = 1,\ j \geq 2 \\ b/i, & \text{for } j = i + 1 \\ 0, & \text{otherwise,} \end{cases}$$

where $a = 50\%$ is the fertility of the second generation, $q = 85\%$ is the factor of geometric decay of fertility and $b = 75\%$ is the transition from the first age group to the second.

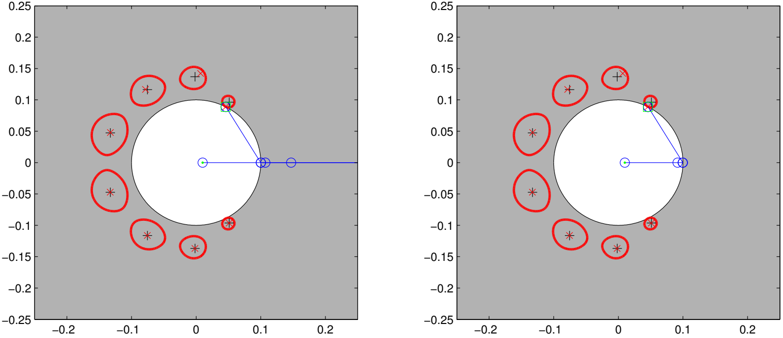
Leslie matrix & D2D for the annulus domain



(a) eD2D, $z_0 = 1.1$

(b) eD2D, $z_0 = 0.01$

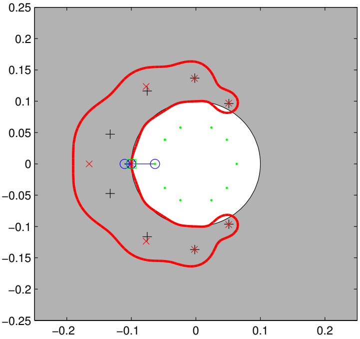
Leslie matrix & D2D for the annulus domain



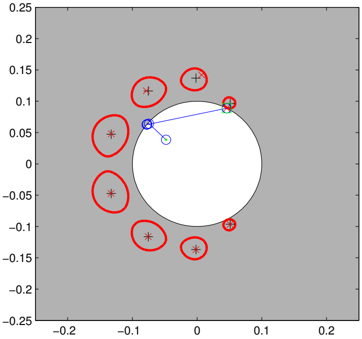
(c) eD2D(g), $z_0 = 0.01$

(d) eD2D_d(g), $z_0 = 0.01$

Leslie matrix & D2D for annulus domain



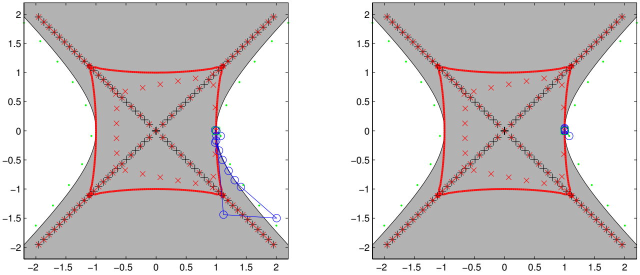
(e) eD2D, $sp = 10$



(f) $\mathrm{eD2D_d(g)}$, $sp = 1$

Twisted matrix & D2D for the hyperbolic domain

A particularly challenging example for the distance to delocalization is the "Twisted" matrix A of dimension $n = 100$ from the EigTool package (an exponentially strong degree of nonnormality and its pseudospectrum grows the fastest around zero).

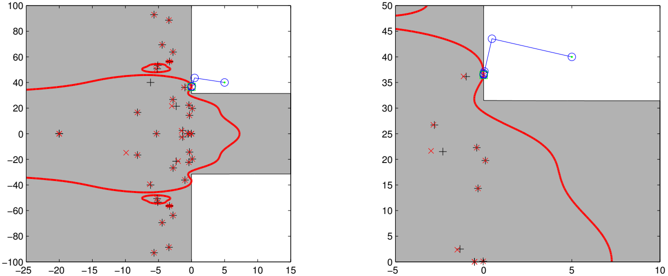


(g) eD2D (h) iD2D

Boeing767 matrix & D2D for the frequency domain ($< 5Hz$)

Let A be the unstable matrix of size $n = 55$ that comes from flutter analysis of the Boeing 767 aircraft (EigTool function `boeing_demo('O')`) with unstable pair of eigenvalues slightly inside the right half-plane that correspond to vibrations with a frequency of approximately $3.15Hz$.

Let us compute the robustness of the unstable oscillations below 5Hz, i.e., the distance to delocalization from the domain Λ_f^+ with a nonstandard basis φ defined by $f(x,y) = -x - y^2 + a^2 + \sqrt{x^2 + (y^2 + a^2)^2}$, where $a = 2\pi \cdot 5 = 31.4159$.



CONCLUSIONS & FURTHER RESEARCH

Conclusions:

Conclusions:

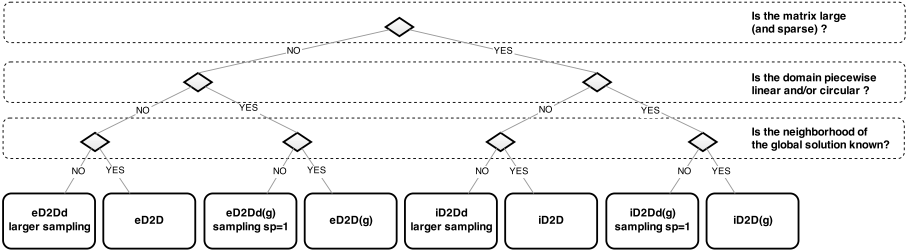
- We introduced matrix nearness problems of delocalization/localization,

Conclusions:

- We introduced matrix nearness problems of delocalization/localization,
- We proposed use of Lyapunov-type domains as a suitable framework for such problems,

Conclusions:

- We introduced matrix nearness problems of delocalization/localization,
- We proposed use of Lyapunov-type domains as a suitable framework for such problems,
- We designed a pseudospectral algorithms for distance to delocalization:



Further research:

Further research:

- Development of methods that assure global optimization (tunneling algorithms with repellers, filtering algorithms, etc.),

Further research:

- Development of methods that assure global optimization (tunneling algorithms with repellers, filtering algorithms, etc.),
- Application to the domains of numerical stability.

Further research:

- Development of methods that assure global optimization (tunneling algorithms with repellers, filtering algorithms, etc.),
- Application to the domains of numerical stability.
- Investigating transient behavior for matrix functions (Kreiss constants).

Further research:

- Development of methods that assure global optimization (tunneling algorithms with repellers, filtering algorithms, etc.),
- Application to the domains of numerical stability.
- Investigating transient behavior for matrix functions (Kreiss constants).
- Computing:
  - real distance to delocalization/localization
  - structured complex distance to delocalization/localization
  - structured real distance to delocalization/localization

Further research:

- Development of methods that assure global optimization (tunneling algorithms with repellers, filtering algorithms, etc.),
- Application to the domains of numerical stability.
- Investigating transient behavior for matrix functions (Kreiss constants).
- Computing:
  - real distance to delocalization/localization
  - structured complex distance to delocalization/localization
  - structured real distance to delocalization/localization
- Development of computational methods for distance to localization using generalized Lyapunov theorem and successive convex approximations,
- Generalization of the delocalization/localization matrix nearness problems to polynomial eigenvalue problems.

Thank you very much for your attention.