M-matrices as a tool for spectral and pseudospectral analysis

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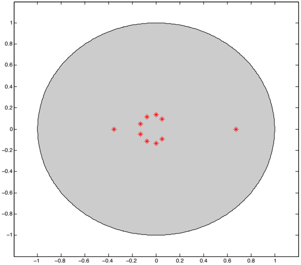
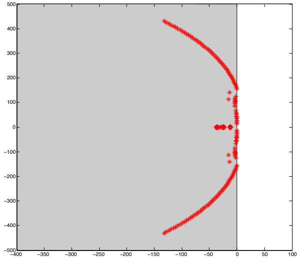
Stability of Dynamical Systems

Stability of linear time invariant dynamical system at equilibrium point depends of the location of the system eigenvalues:



discrete case (d-stability)

$$x_{k+1} = Ax_k, k \in \mathbb{N}$$
$$\Lambda(A) \subseteq \mathcal{B}_1$$



continuous case (c-stability)

$$\dot{x}(t) = Ax(t), t \in \mathbb{R}_0^+$$
$$\Lambda(A) \subseteq \mathbb{C}^-$$

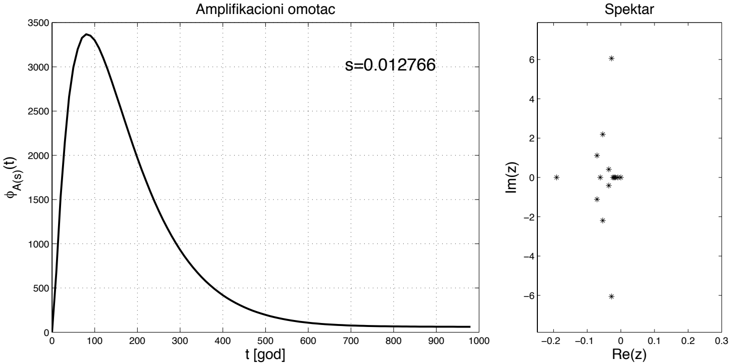
$$\Lambda(A) := \{z \in \mathbb{C} : \det(A - zI) = 0\}$$

Stability of Dynamical Systems

For a continous case a solution is known tobe $x(t) = \exp(tA)x(0)$, so the "deviation" from the equilibria is goverend by the amplification envelope

$$\phi_A(t) := \|e^{tA}\|_p = \sup_{x(0)\neq 0} \frac{\|x(t)\|_p}{\|x(0)\|_p}\ , \quad t \geq 0$$

where $\|\cdot\|_p$ is a matrix norm induced by the vector norm $\|\cdot\|_p$.



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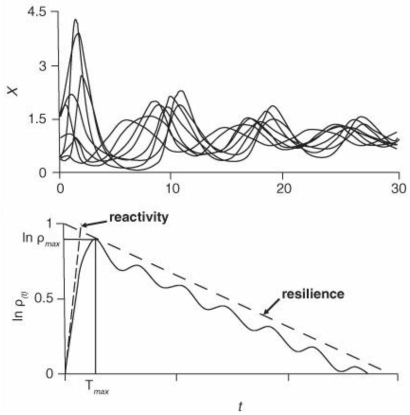

Properties of the amplification envelope:

- return time and norm-invariant r.t.:

$$t_A^{RT} = \min\left\{t > 0 : \phi_A(t) \leq e^{-1}\right\},\ t_A^{NIRT}$$

- resilience:

$$\rho_A = \left[t_A^{NIRT}\right]^{-1}$$

- reactivity:

$$\frac{d^+}{dt}\left[\phi_A(t)\right]\Big|_{t=0}$$

- maximal amplification

$$\phi_A^{\max} = \max_{t\geq 0} \phi_A(t)$$

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- transient behavior $\Longrightarrow$ matrix pseudospectrum

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- resilience $\Longrightarrow$ matrix eigenvalues, i.e. matrix spectrum
- transient behavior $\Longrightarrow$ matrix pseudospectrum
- reactivity $\Longrightarrow$ logarithmic matrix norm (matrix measure)

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Stability via matrix characteristics - resilience

Resilience and return time

Let $A \in \mathbb{R}^{n,n}$ and $x(0) \in \mathbb{R}$ be arbitrary matrix and arbitrary vector. Then for every component $x_i(t)$, $1 \leq i \leq n$, of trajectory $x(t)$ of LTIDS

$$\dot{x}(t) = Ax(t)$$

$\lim_{t\to\infty} x_i(t) = 0$ holds if and only if every eigenvalue λ of the matrix A is in the open left half-plane of $\mathbb{C}$, i.e., $Re(\lambda) < 0$. Therefore, $x^\star = 0$ is an (exponentially) stable equilibrium of LTIDS if and only if its spectral abscissa

$$\alpha_A := \max\{Re(\lambda) \ : \ \lambda \in \Lambda(A)\}$$

is negative. Moreover, the resilience of such a system is $\rho_A = -\alpha_A$, and norm-invariant return time is $t_{NIRT} = -\alpha_A^{-1}$.

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Let $\varepsilon > 0$. The $\varepsilon-$pseudospectrum of a matrix A is

$$\Lambda_\varepsilon(A) = \bigcup_{\|\Delta\|\leq\varepsilon} \Lambda(A+\Delta) = \{z \in \mathbb{C} : \|(zI - A)^{-1}\|^{-1} \leq \varepsilon\}$$

Therefore, ε-pseudospectrum is used to establish spectral properties that are *robust* under matrix perturbations bounded in a given norm by the parameter $\varepsilon > 0$.

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Therefore, ε-pseudospectrum is used to establish spectral properties that are *robust* under matrix perturbations bounded in a given norm by the parameter $\varepsilon > 0$.

It is known that when matrix A is nonnormal, i.e. $AA^* \neq A^*A$, euclidean $\Lambda_\varepsilon(A)$ can be very far from $\Lambda(A) + \{z \in \mathbb{C} : |z| \leq \varepsilon\}$!

Stability via matrix characteristics - transient behavior

Maximal amplification

Let $A \in \mathbb{R}^{n,n}$ and $x(0) \in \mathbb{R}$ be arbitrary matrix and arbitrary vector, and $\|\cdot\|$ given matrix norm induced by a vector norm. If

$$\mathcal{K}(A) := \sup_{Re(z)>0} \frac{Re(z)}{\|(zI - A)^{-1}\|^{-1}} = \sup_{\varepsilon>0} \frac{\alpha_A^\varepsilon}{\varepsilon} < \infty,$$

where

$$\alpha_A^\varepsilon := \max\{Re(z) \ : \ z \in \Lambda_\varepsilon(A)\},$$

denotes $\varepsilon-$pseudospectral abscissa of A, then $x^\star = 0$ is exponentially stable equilibrium of LTIDS and

$$\mathcal{K}(A) \leq \phi_A^{\max} = \sup_{t\geq 0} \|e^{At}\| \leq en\mathcal{K}(A).$$

The value $\mathcal{K}(A)$ is called the Kreiss constant.

Stability via matrix characteristics - reactivity

$$\nu_A = \frac{d^+}{dt}\left[\phi_A(t)\right]\Big|_{t=0} = \nu_A = \lim_{t\searrow 0} \frac{\|e^{At}\| - \|e^{A\cdot 0}\|}{t} = \lim_{t\searrow 0} \frac{\|E + tA\| - 1}{t}.$$

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Matrix measure for max norm

If $\|\cdot\| = \|\cdot\|_\infty$, the reactivity of A is

$$\nu_A = \max_k\left\{Re(a_{kk}) + \sum_{j\neq k}|a_{kj}|\right\},$$

or, equivalently, reactivity is the Geršgorin abscissa

$$\nu_A = \gamma_A := \max\{Re(z)\ :\ z\in\Gamma(A)\},$$

bearing in mind that the set

$$\Gamma(A) := \bigcup_{i\in N}\left\{z\in\mathbb{C}\ :\ |z-a_{i,i}|\leq\sum_{j\neq i}|a_{ij}|\right\}$$

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Matrix measure for norm 1

If $\|\cdot\| = \|\cdot\|_\infty$, the reactivity of A is

$$\nu_A = \max_k\left\{Re(a_{kk}) + \sum_{j\neq k}|a_{jk}|\right\},$$

or, equivalently, reactivity is the column-wise Geršgorin abscissa

$$\nu_A = \gamma_A := \max\{Re(z) \;:\; z\in\Gamma(A^T)\},$$

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Matrix measure for euclidean norm

If $\|\cdot\| = \|\cdot\|_\infty$, the reactivity of A is the spectral abscissa of the Hermitian part of the matrix A

$$\nu_A = \alpha_{H_A} = \max\left\{\lambda \ : \ \lambda \in \Lambda\left(\frac{A + A^*}{2}\right)\right\},$$

or, equivalently, reactivity is the numerical abscissa

$$\nu_A = \omega_A := \max\{Re(z) \ : \ z \in W(A)\},$$

bearing in mind that the set

$$W(A) = \left\{\frac{x^* A x}{x^* x} \ : \ x \in \mathbb{C}^n, \ x \neq 0\right\}$$

is known as the numerical range.

Stability via matrix characteristics - summary

One can determine the properties of equilibrium $x^\star = 0$ of LTIDS by checking the position of an adequate set in $\mathbb{C}^-$ which is quantified by it's abscissa

Property of LTIDS with A	set	abscissa
exponential stability	spectrum $\Lambda(A)$	α_A
$\varepsilon-$robust exponential stability	$\varepsilon-$pseudospectrum $\Lambda_\varepsilon(A)$	α_A^ε
non-reactivity in max norm	Geršgorin set $\Gamma(A)$	γ_A
non-reactivity in norm 1	column-wise Geršgorin set $\Gamma(A^T)$	γ_{A^T}
non-reactivity in Euclidean norm	numerical range $W(A)$	w_A

In addition, the transient behavior in a given norm can be explained by checking the propagation of the corresponding $\varepsilon-$pseudospectrum in the right half-plane of $\mathbb{C}$ which is quantified by the Kreiss constants.

Spectral Matrix Nearness Problems

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Distance to d-instability

For a given $A \in \mathbb{S}_d^{n,n}$ solve

$$\inf_{\hat{A}} \|\hat{A} - A\| \quad \text{s.t. } \hat{A} \notin \mathbb{S}_d^{n,n}$$

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For a given $A \in \mathbb{S}_c^{n,n}$ solve

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K. V., Międ̦lar A., Stolwijk J., On Matrix Nearness Problems: Distance to Delocalization. *SIMAX*, 35(2) 435–460, 2015

Distance to instability

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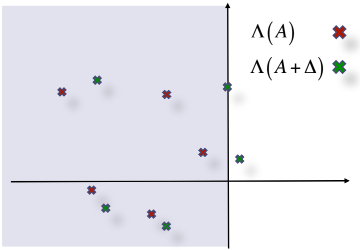

$$\inf \|\Delta\|$$
$$\text{s.t. } \Lambda(A+\Delta) \not\subseteq \mathbb{C}^-$$

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For a given A such that $\Lambda(A) \subseteq \mathbb{C}^-$ solve

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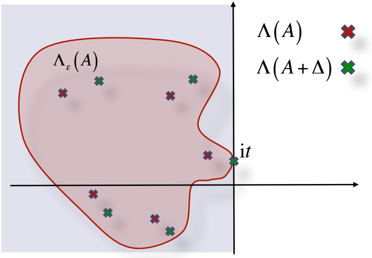

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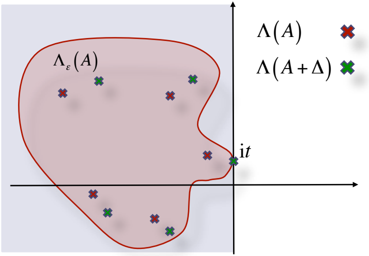

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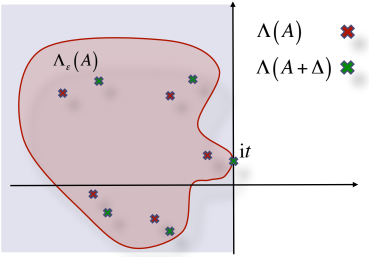

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$$\min \varepsilon$$

$$\text{s.t. } (A - \imath t I)v = \varepsilon u$$

$$(A - \imath t I)^* u = \varepsilon v$$

$$\Delta = \varepsilon u v^*$$

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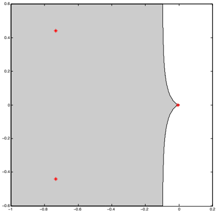

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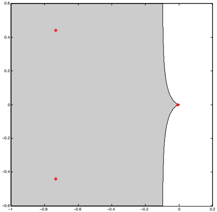
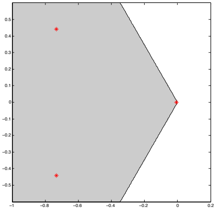


Another damped stability domain that occurs in practice is a wedge around real axis:

the "stable" eigenvalues have complex arguments between $\pi - \theta$ and $\pi + \theta$



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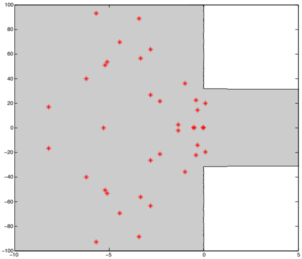
The system is not producing noise of the frequency between a_lHz and a_hHz when the spectrum is either

– in the left half-plane (stable modes), or
– out of the horizontal strips in the right half-plane $[-2\pi a_h, -2\pi a_l]$ and $[2\pi a_l, 2\pi a_h]$ (frequency region of unstable modes)



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- A is nonsingular and A^{-1} is nonegative *(inverse monotone)*
- there exist a vector $w > 0$ such taht $Aw > 0$ *(positive vector)*
- A can be represented as $A = sI - B$, where $B \geq 0$ and $s > \rho(B)$ *(spectral radius)*

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Berman A, Plemmons R., Nonnegative Matrices in the Mathematical Sciences, *Society for Industrial and Applied Mathematics, 1994.*

Definition (few of the many...)

Given arbitrary $A \in \mathbb{R}^{n,n}$ such that

- $a_{ij} \leq 0$ for all i, j in the inex set N, $j \neq i$ *(Z-matrix)*

A is said to be a nonsingular M-matrix if ether one of the following equivalent conditons hold

- A is nonsingular and A^{-1} is nonegative *(inverse monotone)*
- there exist a vector $w > 0$ such taht $Aw > 0$ *(positive vector)*
- A can be represented as $A = sI - B$, where $B \geq 0$ and $s > \rho(B)$ *(spectral radius)*
- the left-most eigenvalue $\mu(A)$ of the matrix A is positive *(Perron-Frobenious)*

(Block) *H*-matrices...

Definition (point-wise form)

Given arbitrary $A \in \mathbb{C}^{n,n}$ its compariosn matrix is $\langle A \rangle = [\alpha_{kj}]$:

$$\langle A \rangle = [\alpha_{kj}] \in \mathbb{R}^{n,n}, \qquad \alpha_{kj} = \begin{cases} |a_{kk}|, & k = j \\ -|a_{kj}|, & k \neq j \end{cases}$$

(Block) *H*-matrices...

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Definition (block comparison matrix)

Given arbitrary $A \in \mathbb{C}^{n,n}$, by $\pi = \{n_j\}_{j=0}^{\ell}$, where nonnegative numbers $n_j,\ j = 1, 2, \ldots, \ell$, satisfy $n_0 := 0 < n_1 < n_2 < \cdots < n_\ell := n$, we denote a *partition* of the index set N.
For arbitrary p-norm, we define the block comparison matrix of $n \times n$ matrix A partitioned into $\ell \times \ell$ blocks by :

$$\left(\langle A \rangle_\pi^{(p)}\right)_{kj} := \begin{cases} \|A_{kk}^{-1}\|_p^{-1} & \text{if} \quad k = j \\ -\|A_{kj}\|_p & \text{if} \quad k \neq j, \end{cases}$$

(Block) *H*-matrices...

Definition

A complex matrix $A \in \mathbb{C}^{n,n}$ is called (block) (nonsingular) H-matrix iff its (block) comparison matrix ($\langle A \rangle_{\pi}^{(p)}$) $\langle A \rangle$ is a (nonsingular) M-matrix.

(Block) *H*-matrices...

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So, in point-wise case, this means:

- $\langle A \rangle$ is nonsingular and $\langle A \rangle^{-1}$ is nonegative *(inverse monotone)*,
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but also iff there exists a postive diagonal matrix $W = diag(w_1, w_2, ..., w_n)$ so that AW is a strictly diagonally dominant (SDD) matrix.

Usefull tools from the M-matrix theory

- For every two Z-matrices $A, B \in \mathbb{R}^{n,n}$,

$$A \leq B \quad \text{imples that} \quad \mu(A) \leq \mu(B),$$

and, thus, if A is a (nonsingular) M-matrix, B is a (nonsingular) M-matrix, too, and

$$A^{-1} \geq B^{-1} \geq 0.$$

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- For any complex matrix A the left-most eigenvalue of a matrix $\langle A \rangle$ is characterized as

$$\mu\left(\langle A \rangle\right) := \inf_{x>0} \max_{i \in N} \frac{(\langle A \rangle x)_i}{x_i} = \inf_{x>0} \max_{i \in N} \left(\langle X^{-1} A X \rangle\right)_i,$$

while

$$\rho(|A|) = \inf_{x>0} \|X^{-1} A X\|_\infty = \inf_{x>0} \max_{i \in N} \left(|X^{-1} A X|\right)_i$$

Usefull tools from the M-matrix theory

- For every partition π and p-norm, if A is block H-matrix, then

$$\|A^{-1}\|_p^{-1} \geq \left\| \left[\langle A \rangle_\pi^{(p)} \right]^{-1} \right\|_p^{-1}.$$

Usefull tools from the M-matrix theory

- For every partition π and p-norm, if A is block H-matrix, then

$$\|A^{-1}\|_p^{-1} \geq \left\| \left[\langle A \rangle_\pi^{(p)} \right]^{-1} \right\|_p^{-1}.$$

- For pointwise partition and $p = \infty$, if A is nonsingular H-matrix, then

$$\|A^{-1}\|_\infty^{-1} \geq \min_{i \in N} x_i^{-1},$$

where $x > 0$ solves $\langle A \rangle x = e$.

- For pointwise partition and $p = 2$, if A is nonsingular H-matrix, then

$$\sigma_n(A) \geq \sigma_n(\langle A \rangle) \geq \min_{i,j \in N} \frac{1}{x_i y_j},$$

where $x > 0$ and $y > 0$ solve $\langle A \rangle x = e$ and $\langle A \rangle^T y = e$.

ON GERŠGORIN-TYPE SPECTRAL LOCALIZATIONS

Geršgorin's Circles...

Given an arbitrary matrix $A = [a_{ij}] \in \mathbb{C}^{n,n}$, its *i-th Geršgorin's disk* is defined by

$$\Gamma_i(A) := \{z \in \mathbb{C} : |z - a_{ii}| \leq r_i(A) := \sum_{j \neq i} |a_{ij}|\}, \quad i = 1, \ldots, n$$

and the union of all these disks, denoted by

$$\Gamma(A) := \bigcup_{i=1}^{n} \Gamma_i(A),$$

is called the *Geršgorin's set* of matrix A.

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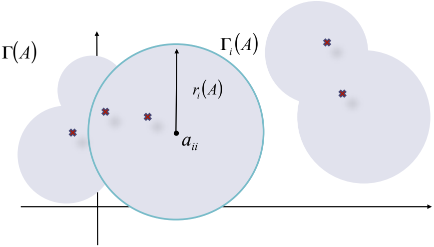
A well-known result of Geršgorin (1931) states that $\Gamma(A)$ contains the spectrum $\sigma(A)$ of matrix A, i.e.

$$\Lambda(A) \subseteq \Gamma(A).$$



Geršgorin's set & SDD matrices

Geršgorin's set is closely related to the class of complex matrices called strictly diagonally dominant (SDD) matrices, defined by the condition

$$|a_{ii}| > r_i(A) := \sum_{j \in N \setminus \{i\}} |a_{ij}|, \text{ for all } i \in N := \{1, 2, \ldots, n\}.$$

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Theorem (Varga R.: *Geršgorin and His Circles*, Springer, 2004)

The following two statements are equivalent:

- *Gerğorin theorem holds true, i.e., for all* $A \in \mathbb{C}^{n,n}$, $\Lambda(A) \subseteq \Gamma(A)$.
- *Every SDD matrix is nonsingular, i.e., if for all* $i \in N$ $|a_{ii}| > r_i(A)$, *then* $\det(A) \neq 0$.

SDD & GDD matrices

Many generalizations of the SDD class have been constructed and used to obtain usefull nonsingularity results and/or localizations of eigevalues:

- doubly SDD (Brauer's ovals of Cassini),
- cycle SDD (Brualdi lemniscate sets),
- S-SDD (CKV sets),
- Ostrowski 1&2 SDD (Ostrowski's sets)
- ...
- GDD (Minimal Geršgorin set)

Geršgorin-type spectral localizations

K. V.: *On general principles of eigenvalue localizations via diagonal dominance*, Advances in Computational Mathematics 41(1) 55-75, 2015

The goal was to provide a unified approach for eigenvalue localizations vagely called of Geršgorin type. To that end the follwing was obtained:

- concept of DD-type and SDD-type classes of matrices,
- Geršgorin-type localization sets,
- Monotonicity principle,
- Equivalence principle,
- Compactness principle
- Isolation Principle,
- Inertia & Radius Principles

Geršgorin-type spectral localizations

Definition

Let $\mathbb{K}$ be a nonempty class of square matrices of an arbitrary size. If $\mathbb{K}$ is such that:

- for any $A \in \mathbb{K}$, diagonal entries of A are nonzero,
- for every $A \in \mathbb{K}$ and every $B \in \mathbb{C}^{n,n}$, if $\langle B \rangle \geq \langle A \rangle$, then $B \in \mathbb{K}$,

then we say that $\mathbb{K}$ is a **diagonally dominant-type**, or briefly **DD-type**, class of matrices.

Here, $\langle A \rangle := [m_{ij}] \in \mathbb{R}^{n,n}$ defined by

$$m_{ij} := \begin{cases} |a_{ii}|, & i = j, \\ -|a_{ij}|, & \text{otherwise.} \end{cases}$$

denotes the **comparison matrix** of the matrix A.

Geršgorin-type spectral localizations

Theorem (Equivalence Principle)

Given a class of square complex matrices of an arbitrary size, denoted by $\mathbb{K}$, for an arbitrary square matrix A, define the set of complex numbers

$$\Theta^{\mathbb{K}}(A) := \{z \in \mathbb{C} \,:\, A - zI \notin \mathbb{K}\}\,.$$

Then, the following two conditions are equivalent:

- *All matrices from $\mathbb{K}$ are nonsingular;*
- *Given an arbitrary square matrix A, the set $\Theta^{\mathbb{K}}(A)$ contains all its eigenvalues, i.e., $\Lambda(A) \subseteq \Theta^{\mathbb{K}}(A)$.*

When $\mathbb{K}$ is a DD-type, the mappping $\Theta^{\mathbb{K}}$ we name as the Geršgorin-type spectral localization.

Principles of Geršgorin-type spectral localizations

Theorem (Monotonicity Principle)

The mapping $\Theta^{\mathbb{K}}$ *is monotone in* $\mathbb{K}$*, i.e. if* $\mathbb{K}_1 \subseteq \mathbb{K}_2$*, then, for an arbitrary square matrix* A*,* $\Theta^{\mathbb{K}_2}(A) \subseteq \Theta^{\mathbb{K}_1}(A)$ *holds.*

Theorem (Isolation Principle)

Given a nonsingular DD-type class of matrices $\mathbb{K}$ *and arbitrary matrix* $A \in \mathbb{C}^{n,n}$*,* $n \geq 2$*, if there exist closed disjoint sets* $U, V \subseteq \mathbb{C}$ *such that for the corresponding Geršgorin-type set* $\Theta^{\mathbb{K}}(A)$

$$\Theta^{\mathbb{K}}(A) = U \cup V,$$

then, the number of eigenvalues and the number of diagonal entries of the matrix A *in the set* U *coincide.*

Principles of Geršgorin-type spectral localizations

For a given class of matrices $\mathbb{K}$ we say that it is:

- **star-shaped**, if for every real $\alpha > 0$, $A \in \mathbb{K}$ implies $\alpha A \in \mathbb{K}$.
- **open**, if for every matrix $A \in \mathbb{K}$, there exists an arbitrary small $\varepsilon > 0$, such that for every matrix $B \in \mathbb{C}^{n,n}$, $|(A - B)_{ij}| < \varepsilon$, for all $i, j \in N$, implies $B \in \mathbb{K}$.
- **SDD-type** class if it is nonempty open star-shaped DD-type class of matrices.

Theorem (Compactness Principle)

Given a nonsingular SDD-type class of matrices $\mathbb{K}$ and an arbitrary matrix $A \in \mathbb{C}^{n,n}$, the corresponding Geršgorin-type set $\Theta^{\mathbb{K}}(A)$ is a compact set in complex plane.

Principles of Geršgorin-type spectral localizations

Let d_+ and d_- denote the number of diagonal entries of A which have positive real part and negative real part, respectively, and denote $\langle A\rangle_{re} := [m_{ij}] \in \mathbb{R}^{n,n}$ defined by

$$m_{ij} := \begin{cases} |\mathrm{Re}\,(a_{ii})|, & i = j, \\ -|a_{ij}|, & \text{otherwise.} \end{cases}$$

Theorem (Inertiia Principle - real case)

Given a nonsingular SDD-type class $\mathbb{K}$, an arbitrary real matrix $A \in \mathbb{R}^{n,n}$, $n \geq 2$, and the corresponding Geršgorin-type set $\Theta^{\mathbb{K}}(A)$, Then, $\langle A\rangle \in \mathbb{K}$ implies $\Theta^{\mathbb{K}}(A) \cap \imath\mathbb{R} = \emptyset$, and, consequently, $in(A) = (d_+, 0, d_-)$.

Theorem (Inertiia Principle - complex case)

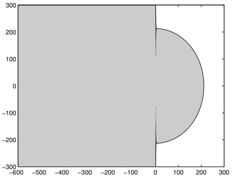
Given a nonsingular SDD-type class $\mathbb{K}$, an arbitrary complex matrix $A \in \mathbb{C}^{n,n}$, $n \geq 2$, and the corresponding Geršgorin-type set $\Theta^{\mathbb{K}}(A)$, Then, $\langle A\rangle_{re} \in \mathbb{K}$ if and only if $\Theta^{\mathbb{K}}(A) \cap \imath\mathbb{R} = \emptyset$, and, consequently, $in(A) = (d_+, 0, d_-)$.

Generalization to Nonlinear Eigenvalue problems

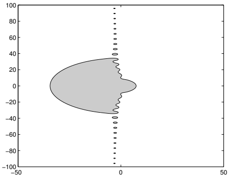
K. V., Gardašević D.,: *On Geršgor-type Localizations for Nonlinear Eigenvalue Problems*, Applied Mathematics and Computation 337, 179-189, 2018

Time delay is 3×3 nonlinear eigenvalue problem from time delay system. The characteristic function $T_{td} \in \mathcal{N}_3(\mathbb{C})$ of a time delay system with a single delay and constant coefficients is $T_{td}(z) = -3zI + A + e^{-z}B$, where:

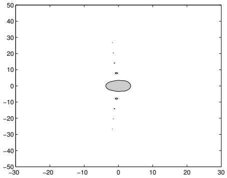
$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -531.6456 & -107.5599 & -3.9852 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1363.7 & -18.7335 & -13.3519 \end{bmatrix}.$$



(a) Set $\Theta^{\mathbb{S}}(T_{td})$



(b) Set $\Theta^{\mathbb{S}_o}(T_{td})$



(c) Set $\Theta^{\mathbb{H}}(T_{td})$

ON GERŠGORIN-TYPE PSEUDOSPECTRAL LOCALIZATIONS

Pseudospectra localization principle

Definition (via resolvent)

Given arbitrary $A \in \mathbb{C}^{n,n}$ and $\varepsilon > 0$

$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} \ : \|(A - zI)^{-1}\|^{-1} \leq \varepsilon\}.$$

Pseudospectra localization principle

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$$\Lambda_\varepsilon(A) = \{z \in \mathbb{C} \ : \|(A - zI)^{-1}\|^{-1} \le \varepsilon\}.$$

Lemma (Localization Principle)

Given $s : \mathbb{C}^{n,n} \to \mathbb{R}$ *such that for an arbitrary matrix* A

$$\|A^{-1}\|^{-1} \ge s(A)$$

holds true, then

$$\Lambda_\varepsilon(A) \subseteq \Theta_\varepsilon^s(A) := \{z \in \mathbb{C} \ : \ s(A - zI) \le \varepsilon\}.$$

Obviously, different bounds (s) give rise to different $\varepsilon-$pseudospectra localization sets. In the following we derive several, generally different, lower bounds for $\|A^{-1}\|^{-1}$, and construct the corresponding $\varepsilon-$pseudospectra localizations.

ε-pseudo Geršgorin sets

This bound is applicable to all matrices, but it is closely related to SDD matrices.

Lemma

Given an arbitrary matrix $A \in \mathbb{C}^{n,n}$,

$$\|A^{-1}\|_{\infty}^{-1} \geq s_{\infty}(A) := \min_{i \in N} \left(|a_{ii}| - r_i(A)\right)$$

holds true.

Theorem (ε-pseudo Geršgorin sets)

For an arbitrary matrix $A \in \mathbb{C}^{n,n}$,

$$\Lambda_{\varepsilon}^{(\infty)}(A) \subseteq \Gamma_{\varepsilon}(A) := \bigcup_{i \in N} \{z \in \mathbb{C} \,:\, |a_{ii} - z| \leq r_i(A) + \varepsilon\}\,.$$

ε-pseudo Geršgorin sets

On the other hand, in case of 2-norm we have that

Lemma

Given an arbitrary matrix $A \in \mathbb{C}^{n,n}$,

$$\|A^{-1}\|_2^{-1} \geq s_2(A) := \min_{i \in N} \left\{ |a_{ii}| - \frac{r_i(A) + r_i(A^T)}{2} \right\}$$

holds true.

Theorem (ε-pseudo Geršgorin sets)

For an arbitrary matrix $A \in \mathbb{C}^{n,n}$,

$$\Lambda_\varepsilon^{(2)}(A) \subseteq \Gamma_\varepsilon^{(2)}(A) := \bigcup_{i \in N} \left\{ z \in \mathbb{C} \,:\, |a_{ii} - z| \leq \frac{r_i(A) + r_i(A^T)}{2} + \varepsilon \right\}.$$

ε-pseudo Brauer set

This bound is closely related to *doubly strictly diagonally dominant (dSDD)* matrices.

Lemma

Given an arbitrary matrix $A \in \mathbb{C}^{n,n}$,

$$\|A^{-1}\|_\infty^{-1} \geq \min_{i \neq j : |a_{ii}| + r_j(A) \neq 0} \frac{|a_{ii}||a_{jj}| - r_i(A) r_j(A)}{|a_{ii}| + r_j(A)}$$

holds true, where the minimum over an empty set is defined to be zero.

Theorem (ε-pseudo Brauer set)

Given an arbitrary matrix $A \in \mathbb{C}^{n,n}$*, the set*

$$\mathcal{B}_\varepsilon(A) := \bigcup_{i \neq j} \left\{ z \in \mathbb{C} \,:\, |a_{ii} - z| \big(|a_{jj} - z| - \varepsilon \big) \leq r_j(A) \big(r_i(A) + \varepsilon \big) \right\}$$

localizes the ε*-psudospectrum of matrix* A*, i.e.,* $\Lambda_\varepsilon^{(\infty)}(A) \subseteq \mathcal{B}_\varepsilon(A)$.

ε-pseudo CKV set

Another interesting result of this kind can be obtained in connection with $S-$SDD matrices.

To simplify notation, let $T := \{(i,j) \in S \times \overline{S} \,:\, |a_{ii}| > r_i^S(A) \text{ and } |a_{jj}| > r_j^{\overline{S}}(A)\}$, and, for $(i,j) \in T$, define

$$\alpha_{ij}^S(A) := \frac{\left(|a_{ii}| - r_i^S(A)\right)\left(|a_{jj}| - r_j^{\overline{S}}(A)\right) - r_i^{\overline{S}}(A) r_j^S(A)}{\max\{|a_{ii}| - r_i^S(A) + r_j^S(A),\ |a_{jj}| - r_j^{\overline{S}}(A) + r_i^{\overline{S}}(A)\}}.$$

Lemma

Given an arbitrary matrix $A \in \mathbb{C}^{n,n}$ and an arbitrary set of indices $S \subseteq N$,

$$\|A^{-1}\|_\infty^{-1} \geq \min\left\{ \min_{i \in S}\left(|a_{ii}| - r_i^S(A)\right), \min_{j \in \overline{S}}\left(|a_{jj}| - r_j^{\overline{S}}(A)\right), \min_{(i,j) \in T} \alpha_{ij}^S(A) \right\}$$

holds true.

ε-pseudo CKV set

Theorem (ε-pseudo CKV sets)

Given an arbitrary matrix $A \in \mathbb{C}^{n,n}$ and an arbitrary set of indices $S \subseteq N$, ε-pseudospectrum of A is localized by the set $\mathcal{C}_\varepsilon(A)$, i.e.

$$\Lambda_\varepsilon^{(\infty)}(A) \subseteq \mathcal{C}_\varepsilon^S(A) := \Gamma_\varepsilon^S(A) \cup \Gamma_\varepsilon^{\overline{S}}(A) \cup V_\varepsilon^S(A) \cup V_\varepsilon^{\overline{S}}(A),$$

where

$$\Gamma_\varepsilon^S(A) := \bigcup_{i\in S} \left\{ z \in \mathbb{C} : |z - a_{ii}| \leq r_i^S(A) + \varepsilon \right\}, \quad \Gamma_\varepsilon^{\overline{S}}(A) := \bigcup_{j\in \overline{S}} \left\{ z \in \mathbb{C} : |z - a_{jj}| \leq r_j^{\overline{S}}(A) + \varepsilon \right\},$$

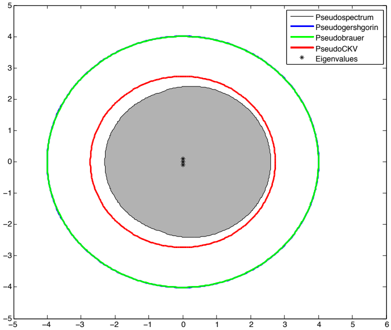
$$V_\varepsilon^S(A) := \bigcup_{i\in S, j\in \overline{S}} \left\{ z \in \mathbb{C} : (|z - a_{ii}| - r_i^S(A) - \varepsilon)(|z - a_{jj}| - r_j^{\overline{S}}(A)) \leq r_i^{\overline{S}}(A)\big(r_j^S(A) + \varepsilon\big) \right\},$$

$$V_\varepsilon^{\overline{S}}(A) := \bigcup_{i\in S, j\in \overline{S}} \left\{ z \in \mathbb{C} : (|z - a_{ii}| - r_i^S(A))(|z - a_{jj}| - r_j^{\overline{S}}(A) - \varepsilon) \leq \big(r_i^{\overline{S}}(A) + \varepsilon\big) r_j^S(A) \right\}.$$

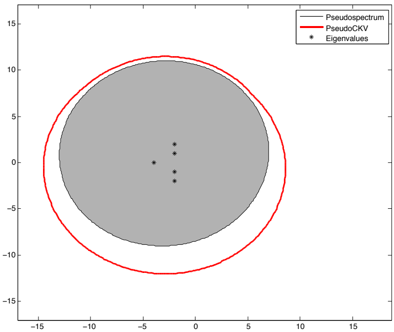
Numerical examples

$$M_1 = \begin{bmatrix} 0 & 1 & 2 \\ -0.01 & 0 & 3 \\ 0 & 0 & 0 \end{bmatrix}, \quad \text{and} \quad M_2 = \begin{bmatrix} -4 & 1 & 1 & 1 & 10^5 \\ & -2+\imath & 1 & 1 & 1 \\ & & -2-\imath & 1 & 1 \\ & & & -2+2\imath & 1 \\ & & & & -2-2\imath \end{bmatrix}.$$

Localization sets Γ_ε, $\mathcal{B}_\varepsilon$ and $\mathcal{C}_\varepsilon^{\{1,2\}}$ for ε-pseudospectrum of matrix M_1 (left) and set $\mathcal{C}_\varepsilon^{\{1,2\}}$ for matrix M_2 (right).



(d) $\varepsilon/\|M_1\|_\infty = 0.33$

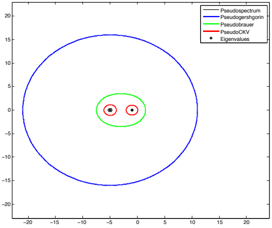


(e) $\varepsilon/\|M_2\|_\infty = 10^{-8}$

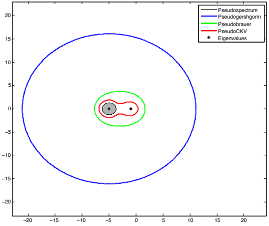
Numerical examples

$$M_3 = \begin{bmatrix} -1 & 1 & & \\ & -5 & & \\ & 15 & -5 & 1 \\ & & & -1 \end{bmatrix}$$

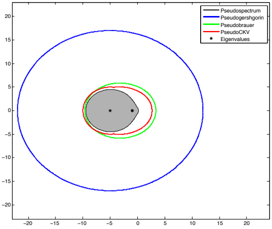
Localization sets Γ_ε, $\mathcal{B}_\varepsilon$ and $\mathcal{C}_\varepsilon^{\{1,2\}}$ for ε-pseudospectrum of the matrix M_3 for three different values of ε.



(a) $\varepsilon/\|M_3\|_\infty = 10^{-4}$



(b) $\varepsilon/\|M_3\|_\infty = 10^{-3}$

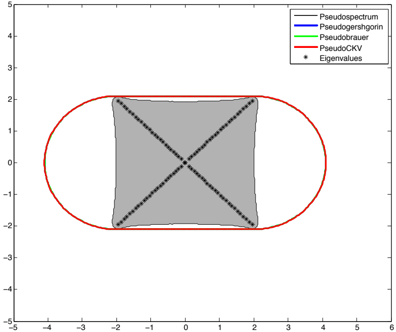


(c) $\varepsilon/\|M_3\|_\infty = 10^{-2}$

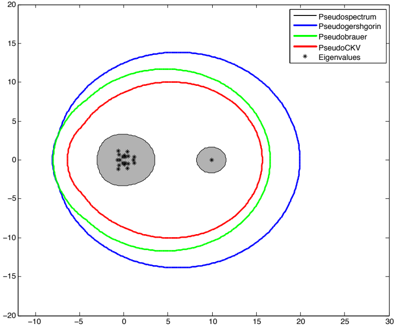
Numerical examples

$$M_4 = \begin{bmatrix} s_1 & 1 & & & -1 \\ -1 & s_2 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & s_{n-1} & 1 \\ 1 & & & -1 & s_n \end{bmatrix}, \; s_j := 2\sin\frac{2\pi}{n}, \quad \text{and } M_5 = \texttt{hess(rand(30))}.$$

Localization sets Γ_ε, $\mathcal{B}_\varepsilon$ and $\mathcal{C}_\varepsilon^{\{1,\dots 15\}}$ for ε-pseudospectrum of the matrix M_4 ($n = 100$) and set $\mathcal{C}_\varepsilon^{\{1,\dots 4\}}$ for the matrix M_5 (right).



(a) $\varepsilon / \|M_4\|_\infty = 10^{-2}$



(b) $\varepsilon / \|M_5\|_\infty = 10^{-2}$

LOWER BOUNDS FOR DISTANCE TO INSTABILITY

Lower bounds for distance to instability

Obviously, pseudospectral localization sets can be used to obtain the bounds for distance to instability.

Theorem

Let $A \in \mathbb{C}^{n,n}$, such that $Re(a_i i) < 0$, for all $i \in N$. If $\langle A \rangle_{re}$ is an SDD matrix, $s_\infty\big(\langle A \rangle_{re}\big) > 0$ and

$$\Lambda_\varepsilon^{(\infty)}(A) \subset \Gamma_\varepsilon(A) \subset \mathbb{C}^- \quad \textit{for all} \quad 0 < \varepsilon < s_\infty\big(\langle A \rangle_{re}\big).$$

Theorem

Let $A \in \mathbb{C}^{n,n}$, such that $Re(a_{ii}) < 0$, for all $i \in N$. If $\frac{\langle A \rangle_{re} + \langle A \rangle_{re}^T}{2}$ is an SDD matrix, $s_2\big(\langle A \rangle_{re}\big) > 0$ and

$$\Lambda_\varepsilon^{(2)}(A) \subset \Gamma_\varepsilon^{(2)}(A) \subset \mathbb{C}^- \quad \textit{for all} \quad 0 < \varepsilon < s_2\big(\langle A \rangle_{re}\big).$$

Lower bounds for distance to instability

K. V., Cvetković Lj., Šanca E.,: *From pseudospectra of diagonal blocks to spectrum of the full matrix*, Journal of Computational and Applied Mathematics (submitted 2019)

Theorem

Let $A \in \mathbb{C}^{n,n}$, such that $Re(a_{ii}) < 0$, for all $i \in N$. If $\langle A \rangle_{re}$ is an $H-$matrix, then for all $1 \leq p \leq +\infty$, the p-norm distance to instability $\delta_p(A)$ is bounded by $\|\left(\langle A \rangle_{re}\right)^{-1}\|_p^{-1} > 0$:

$$\delta_p(A) \geq \|\left(\langle A \rangle_{re}\right)^{-1}\|_p^{-1} > 0.$$

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$$\delta_p(A) \geq \| \big(\langle A \rangle_{re} \big)^{-1} \|_p^{-1} > 0.$$

- $\delta_\infty(A) = \min_{t \in \mathbb{R}} \|(A - \imath t I)^{-1}\|_\infty^{-1} \geq \min_{i \in N} x_i^{-1}$, where $\langle A \rangle_{re} x = e$;

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- $\delta_2(A) = \min_{t\in\mathbb{R}} \sigma_n(A - \imath t I) \geq \sigma_n(\langle A \rangle_{re})$;
- If A is stable, then its distance to insability in 2-norm is always bounded by

$$\delta_2(A) = \min_{t\in\mathbb{R}} \sigma_n(A - \imath t I) \geq \sigma_n(\langle T \rangle_{re}),$$

where T is (complex) Schur form of A.

(PSEUDO)SPECTRAL LOCALIZATIONS FOR PARTITIONED MATRICES

Using the block Minimal Geršgorin set for pseudospectra

K. V., Cvetković Lj., Šanca E.,: *From pseudospectra of diagonal blocks to spectrum of the full matrix*, Journal of Computational and Applied Mathematics (submitted 2019)

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Theorem

Let $A \in \mathbb{C}^{n,n}$ be an arbitrary matrix, π an arbitrary partition of its index set and $p \geq 1$. Then,

$$\Lambda_{\varepsilon}^{(p)}(A) \subseteq \mathcal{M}_{\pi,\varepsilon}^{(p)}(A),$$

where

$$\mathcal{M}_{\pi,\varepsilon}^{(p)}(A) := \left\{z \in \mathbb{C} : \|(\langle A - zI\rangle_{\pi}^{(p)})^{-1}\|_p^{-1} \leq \varepsilon\right\} \bigcup \left\{z \in \mathbb{C} : \mu(\langle A - zI\rangle_{\pi}^{(p)}) \leq 0\right\},$$

where μ is the leftmost eigenvalue of $Z-$matrix $\langle A - zI\rangle_{\pi}^{(p)}$.

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where μ is the leftmost eigenvalue of $Z-$matrix $\langle A - zI \rangle_{\pi}^{(p)}$.

Consequently, for $p = 2$ and $\varepsilon = 0$, and partition π in 2×2 blocks we have that

$$\Lambda(A) \subseteq \mathcal{M}_{\pi,\varepsilon}^{(p)}(A) = \{ z \in \mathbb{C} \ : \ \sigma_n(A_{11} - zI)\sigma_n(A_{22} - zI) \leq \|A_{12}\|_2 \|A_{21}\|_2 \} \, .$$

Using the block Minimal Geršgorin set for pseudospectra

$$A = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix},$$

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Theorem

Given a matrix $A \in \mathbb{C}^{n,n}$ partitioned as above for an arbitrary $\varepsilon \geq 0$, denote

$$f(\varepsilon) := \sqrt{(\varepsilon + \|A_{12}\|_2)(\varepsilon + \|A_{21}\|_2)}.$$

Then, the following inclusion holds:

$$\Lambda_{\varepsilon}^{(2)}(A) \subseteq \Lambda_{f(\varepsilon)}^{(2)}(A_{11}) \cup \Lambda_{f(\varepsilon)}^{(2)}(A_{22}).$$

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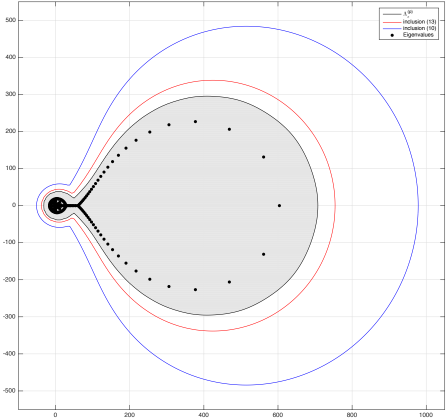
Grammont L., Largillier A.,: *On* $\varepsilon-$*pseudospectra and stability radii*, Journal of Computational and Applied Mathematics 147, 453–469, 2002

Using the block Minimal Geršgorin set for pseudospectra

$$M_2 = \begin{bmatrix} F & 10^{-4}ee^T \\ 10^{-1}ee^T & F \end{bmatrix},$$

where F is the Frankel matrix of size $n = 100$ and e vector of all ones.



SUMMARY

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Using the technique of $M-$matrices we have

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- dervied bounds for distance to instability (pointwise and block),
- dervied bounds for distance to delocalization (pointwise and block).

Thank you very much for your attention.